64 The Shapley Value

1 Introduction

The purpose of this chapter is to present an important solution concept for cooperative games, due to Lloyd S. Shapley (Shapley (1953)). In the first part, we will be looking at the transferable utility (TU) case, for which we will state the main theorem and study several examples. Afterwards, we will extend the axiomatic construction to the non-transferable utility (NTU) case.

2 The Shapley Value in the TU Case

2.1 A First Approach

Let N be a finite set of players and $n = |N|$. A game is a mapping $v\colon 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. For S in 2^N (i.e., $S \subset N$), $v(S)$ may be interpreted as the *worth* of coalition S, i.e. what the players belonging to S can get together by coordinating their efforts. This models a game with transferable utility (or with side payments), i.e., where coalised players may reallocate the total utility within the coalition: it is sufficient to map every coalition to a single number, the coalition's total utility.

The unanimity game U_T associated with the coalition $T \subset N$ is defined by:

$$U_T(S) = \begin{cases} 1, & \text{if } S \supset T; \\ 0, & \text{otherwise.} \end{cases}$$

Given a set of players N, denote by $G(N)$ the set of all possible games with players in N. Let $E = \mathbb{R}^N$ be the space of payoff vectors and for $x \in E$ denote by $x(S)$ the sum $\sum_{n \in S} x_n$. We may then define a value as a mapping $\varphi\colon G(N) \to E$ such that:

$$\forall v \in G(N), (\varphi v)(N) = v(N) \qquad \text{(a)}$$

$$\forall v, w \in G(N), \varphi(v + w) = \varphi v + \varphi w \qquad \text{(b)}$$

$$\forall T \subseteq N, \forall \alpha \in \mathbb{R}, \varphi(\alpha U_T)_i = \begin{cases} \alpha/|T|, & \text{if } i \in T; \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(c)}$$

Axioms (a) and (b) are the standard ones of efficiency and additivity, whereas axiom (c) is equivalent to the axioms: (i) neutral to permutation

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and (ii) null player. It is remarkable that no further conditions are required to determine the value uniquely as in the following (Shapley (1953)):

THEOREM 1 *For each N, there exists a unique value function; this value is given by*

$$(\varphi v)_i = \frac{1}{n!} \sum_{\prec} [v\{j | j \preceq i\} - v\{j | j \prec i\}],$$

where the sum extends over all total orders on the player set.

Before giving the proof, let us observe that the intuitive interpretation of the formula is the following: when a player joins a coalition, it may modify the worth of the coalition; the Shapley value gives to each player his average marginal contribution to the worth of all possible coalitions.

Proof It is easily seen that the function defined above is a value. To prove its uniqueness, it suffices to show that the above games $U_T (T \neq \emptyset, T \subseteq N)$ form a basis. Since their number equals the dimension of $G(N)$, it suffices to show they are linearly independent. Suppose they are not:

$$\exists(\alpha_i) \text{ such that } \sum \alpha_i U_{T_i} = 0 \text{ and } \alpha_j \neq 0 \text{ for some } j. \tag{1}$$

Among the subsets T_i such that $\alpha_i \neq 0$, there exists at least one coalition, say T_1, with a minimum number of players. Then rearranging (1):

$$U_{T_1} = -(1/\alpha_1) \sum_{j>1} \alpha_j U_{T_j};$$

yet $U_{T_1}(T_1) = 1$ and $U_{T_j}(T_1) = 0$ for $j > 1$ because in this case $T_j \not\subset T_1$.

Hence any game may be written as a linear combination of the unanimity games, and by axiom *c*), a value is uniquely determined on these games. QED

2.2 Examples

We shall now examine some examples to underline the differences between the Shapley value and another solution concept, the core.

Example 1 Majority game of 3 players.

$N = \{1, 2, 3\}$, $v(S) = 1$ if $|S| \geqslant 2$ and 0 otherwise. The game is symmetric; player i changes the worth of the coalition that precedes him if he is in position 2, which happens for two different orders. By either argument, $\varphi v = (1/3, 1/3, 1/3)$. On the other hand the core is empty, since there are always two players who can form a coalition and share what the third player gets.

Example 2 Market with one seller and two buyers.

$N = \{1, 2, 3\}$, $v(\{1, 2, 3\}) = v(\{1, 2\}) = v(\{1, 3\}) = 1$ and $v(S) = 0$ otherwise (1 is the seller). Player 2 (or player 3) changes the worth of the coalition that precedes him if player 1 is first and he is second, while player 1 contributes to the coalition as soon as he is not in first: $\varphi v = (2/3, 1/6, 1/6)$. Obviously (1, 0, 0) is in the core, and nothing else is, because any other outcome could be blocked by a coalition of player 1 with one of the other players.

Example 3 A weighted voting game.

$N = \{1, 2, 3, 4\}$, with weights (2, 1, 1, 1); the total weight is 5 and a majority of 3 wins. Player 1 is pivotal in position 2 or 3 (1 chance out of 2), while players 2, 3, and 4 are in symmetric positions; therefore $\varphi v = (1/2, 1/6, 1/6, 1/6)$. Once again the core is empty, since any outcome can be improved upon by the three players who get the least. Note that whereas the large player (player 1) has only 40% of the vote, he gets half the value.

Example 4 Another weighted voting game.

$N = \{1, 2, 3, 4, 5\}$, with weights (3, 3, 1, 1, 1); the core is empty as before, and we have $\varphi v = (3/10, 3/10, 2/15, 2/15, 2/15)$. In this case the large players' value is *less* than their proportion of the vote; thus players 3, 4 and 5 would get less (1/3 instead of 2/5) if they were to unite into a single player with weight 3.

Example 5 Market game with 1,000,000 left gloves and 1,000,001 right gloves—one glove per player.

2,000,001 players, $v(N) = 1{,}000{,}000$. In this case the core has a single element, where the left glove owners get 1 (pair), and the right glove owners get 0. The Shapley value, on the other hand, assigns a total of 500,428 to the left glove owners and a total of 499,572 to the right glove owners.

2.3 Other Characterizations

2.3.1 The Potential

By theorem 1 it is possible to define the Shapley value through the marginal contributions of players: namely the value of a game may be seen as the vector of the players' "expected payoffs" as their expected marginal contributions to coalitions (with the appropriate interpretation, axiom (b) may be seen as an expected utility property). This idea has led to another approach based upon the "potential" of a game. Let us define in a general way the marginal contributions: map every game (N, v) to a real number $P(N, v)$, called the *potential* of the game, and let player i's marginal contribution be: $P(N, v) - P(N \setminus \{i\}, \text{“}v|N \setminus \{i\}\text{”})$. Then one could reasonably require that these marginal contributions satisfy an efficiency condition, i.e. add up to $v(N)$ for all players in N. It is clear inductively that this condition (with an appropriate definition of "$v|N \setminus \{i\}$") determines a unique potential function; moreover it has been shown that it leads precisely to the Shapley value (Hart and Mas-Colell (1989)).

2.3.2 The Monotonicity Principle

Alternative axiomatizations have been put forward. For instance (Young (1985)), it is possible to replace the additivity axiom and the null player axiom by some requirement related to the monotonicity of the value. More precisely, define φ to be an *allocation procedure* if it maps every game to a point in $\mathbb{R}^N$ and is efficient. The procedure φ is *symmetric* (anonymous) if for all permutations π of N, $\varphi_{\pi i}(\pi v) = \varphi_i(v)$, where $\pi v(S) = v(\pi^{-1}(S))$ for all S. The procedure φ satisfies *strong monotonicity* if:

$$\forall \text{ games } v, w\ \forall i \in N, (\forall S \subset N, v^i(S) \geqslant w^i(S)) \Rightarrow (\varphi_i(v) \geqslant \varphi_i(w)),$$

where $v^i(S) = v(S \cup \{i\}) - v(S)$.

In words, strong monotonicity means that the payoff to a player depends only on his marginal contributions—and monotonically. The result is then as follows:

THEOREM 2 *The Shapley value is the unique symmetric allocation procedure that is strongly monotonic.*

2.3.3 A Smaller Class of Games

The axiomatization of the Shapley value requires the application of the value axioms to all games. Yet, it is possible (Neyman (1989)) to derive the Shapley value of any given game v by applying the axioms to a smaller class of games, namely the additive group generated by the subgames of v, which yields a stronger characterization of the Shapley value. Given a game (N, v), and a coalition $S \subset N$, define the subgame v_S as the mapping: $2^N \to \mathbb{R}$ such that $v_S(T) = v(S \cap T)$; $v(S)$ may be viewed as the restriction of v to the subsets of S. Denote by $G(v)$ the additive group generated by the subgames of v:

$$G(v) = \{w \in G(N) | w = \sum k_i v_{S_i} \text{ with } k_i \text{ integers and } S_i \text{ coalitions}\}$$

If Q is a subset of $G(N)$, we say that a map Ψ: $Q \to \mathbb{R}^N$ obeys the *null player axiom* if:

$$\forall v \in Q, \forall i \in N, (v(S \cup \{i\}) = v(S), \forall S \subset N) \Rightarrow (\Psi_i(v) = 0)$$

The extension of the other axioms to a subset of $G(N)$ is straightforward. Then:

THEOREM 3 *Let $v \in G$. If a map Ψ from $G(v)$ into $\mathbb{R}^N$ is efficient, additive, and symmetric, and obeys the null player axiom, then it is the Shapley value.*

Note that, in this case too, it is possible to replace the additivity and null player axioms by strong monotonicity.

3 The Shapley Value in the NTU Case

If x, $y \in \mathbb{R}^N$, then we write $x \geqslant y$ if $x_i \geqslant y_i$ for all i. A set A in $\mathbb{R}^N$ is said to be *comprehensive* if $x \in A$ and $x \geqslant y$ implies $y \in A$. A convex set C in $\mathbb{R}^N$ is said to be *smooth* if it has a unique supporting hyperplane at each point of its frontier ∂C. An *NTU game* is a function V that assigns to each coalition S a convex comprehensive non-empty proper subset $V(S)$ of $\mathbb{R}^S$, such that:

1. $V(N)$ is smooth,
2. If x, $y \in \partial V(N)$ and $x \geqslant y$, then $x = y$,
3. $\forall S \subset N, \exists x \in \mathbb{R}^N$ s.t. $V(S) \times \{0^{N \setminus S}\} \subset V(N) + x$,

where $0^{N\setminus S}$ is the 0-vector in $\mathbb{R}^{N\setminus S}$. The interpretation of the NTU case is that, since no side payments are allowed, there is no possible reallocation. Thus to evaluate the worth of a coalition one has to take into account the payoffs of all players belonging to the coalition; maximizing the worth of a coalition is now a "multi-criterion" problem, in the currently popular jargon. Condition 2 says that the frontier of the grand coalition payoff-set contains only strict Pareto-optima and Condition 3 can be thought of as an extremely weak kind of monotonicity. If v is a TU game, then the NTU game V *corresponding* to v is defined by:

$$V(S) = \left\{x \in \mathbb{R}^S | \sum_{i \in S} x^i \leqslant v(S)\right\};$$

thus we can speak of an NTU *unanimity game* as one corresponding to a TU unanimity game.

We still need to define the Shapley correspondence; the idea (Shapley 1969) is to associate a TU game with every NTU game and comparison vector, with the worth of a coalition S being the best it can get in terms of the comparison vector. More precisely, let V be an NTU game and $\lambda \in \mathbb{R}^N$ a comparison vector (i.e. $\lambda^i > 0, \forall i$). Define an auxiliary TU game v_λ as follows:

$$v_\lambda(S) = \sup\{\langle \lambda^S, x\rangle | x \in V(S)\};$$

the game v_λ is well defined if the supremum is finite for all S. A *Shapley value* of V is a point x in the closure $cl(V(N))$ of $V(N)$ such that for some λ, v_λ is well defined, and the vector $(\lambda^i x^i)$ is the Shapley value of v_λ. Now if Γ is the set of all NTU games with at least one Shapley value, the correspondence from Γ to $\mathbb{R}^N$ that assigns to every game V in Γ the set $\Lambda(V)$ of its Shapley values is the *Shapley correspondence*.

Define a *value correspondence* as a correspondence $\Phi\colon \Gamma \to \mathbb{R}^N$ satisfying the following axioms:

a. $\forall V \in \Gamma, \Phi(V) \neq \emptyset$;

b. $\forall V \in \Gamma, \Phi(V) \subset \partial V(N)$;

c. $\forall U, V \in \Gamma, \Phi(U + V) \supset (\Phi(U) + \Phi(V)) \cap \partial(U + V)(N)$;

d. For all unanimity games $U_T, \Phi(U_T) = \{\Pi_T/|T|\}$;

e. $\Phi(clV) = \Phi(V)$;

f. $\forall \lambda \in \mathbb{R}^N, \lambda > 0, \Phi(\lambda V) = \lambda\Phi(V)$;

g. $\forall V, W \in \Gamma$, if $V(N) \subset W(N)$ and $V(S) = W(S)$ for $S \neq N$, then $\Phi(V) \supset F(W) \cap V(N)$.

Axiom (a) is non-emptiness; axiom (b), efficiency, says that all values are Pareto optimal; axiom (c) says that if y and z are values of V and W and if $y + z$ is Pareto optimal in $V + W$, then it is a value of $V + W$; axiom (d) determines the values of the unanimity games (the values are unique); axiom (e) is closure invariance; axiom (f) is scale covariance; axiom (g) is the well-known independence of irrelevant alternatives (I.I.A.), and it says that a value y of a game W remains a value when one removes outcomes other than y from the set $W(N)$ of all feasible outcomes, without changing the $W(S)$ for $S \neq N$. We now have (Aumann (1985)):

THEOREM 4 *There is a unique value correspondence, and it is the Shapley value.*

It is noteworthy that removing I.I.A. (Axiom (g)) is not too damaging, as the following holds (Aumann (1985)):

THEOREM 5 *The Shapley correspondence is the maximal correspondence among those satisfying axioms* (a) *through* (f) (*i.e. if* Φ *satisfies axioms* (a) *through* (f), *then* $\Phi(V) \subset \Lambda(V)$ *for all games* V *in* Γ).

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