65 Spaces of Measurable Transformations[1]

By a *space* we shall mean a measurable space, i.e. an abstract set together with a σ-ring of subsets, called *measurable* sets, whose union is the whole space. The *structure* of a space will be the σ-ring of its measurable subsets. A *measurable transformation* from one space to another is a mapping such that the inverse image of every measurable set is measurable.

Let X and Y be spaces, F a set of measurable transformations from X into Y, and $\phi_F: F\times X\rightarrow Y$ the natural mapping defined by $\phi_F(f, x)=f(x)$. A structure R on F will be called *admissible* if ϕ_F, considered as a mapping from the product space $(F, R)\times X$ into Y, is a measurable transformation.[2] It may not be possible to define an admissible structure on F; if it is, F itself will also be called *admissible*. We are concerned with the problem of characterizing, for given X and Y, the admissible sets F and the admissible structures R on the admissible sets.

The following three theorems may be established fairly easily:

THEOREM A. *A set consisting of a single measurable transformation is admissible.*

THEOREM B. *A subset of an admissible set is admissible. Indeed, if $G\subset F$, R is an admissible structure on F, and R_G is the subspace structure on G induced*[3] *by R, then R_G is admissible on G.*

THEOREM C. *The union of denumerably many admissible sets is admissible. Indeed, if $F=\bigcup_{i=1}^{\infty} F_i$ and $R_1, R_2, \cdots$ are admissible structures on $F_1, F_2, \cdots$ respectively, then the structure R on F generated by the members of all the R_i is admissible on G.*

Much more can be said if X and Y are assumed to be *separable*, i.e. to have countably generated structures.[4] To state our theorems in this case we first define the concept of Banach class, closely related to that of Baire class. Let $\mathfrak{A}$ be an arbitrary class of meas-

[1] The author is much indebted to Professor P. R. Halmos, who suggested a number of significant improvements in the complete version of this note.

[2] (F, R) is the space whose underlying abstract set is F and whose structure is R.

[3] R_G consists of all intersections of G with members of R.

[4] The term is used by analogy with its topological use. We will also use the term "separable structure," meaning a countably generated structure.

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urable subsets of X. For each denumerable ordinal number $\alpha \geqq 1$, we define classes $P_\alpha(\mathfrak{A})$ and $Q_\alpha(\mathfrak{A})$ inductively as follows: $Q_1(\mathfrak{A})$ consists of all denumerable unions of members of $\mathfrak{A}$, and $P_1(\mathfrak{A})$ consists of all complements of members of $Q_1(\mathfrak{A})$; supposing $Q_\beta(\mathfrak{A})$ and $P_\beta(\mathfrak{A})$ to have been defined for all $\beta < \alpha$, we define $Q_\alpha(\mathfrak{A}) = Q_1(\bigcup_{\beta<\alpha} P_\beta(\mathfrak{A}))$ and $P_\alpha(\mathfrak{A}) = P_1(\bigcup_{\beta<\alpha} P_\beta(\mathfrak{A}))$. $Q_\alpha(\mathfrak{A}) \cup P_\alpha(\mathfrak{A})$ is the set of all subsets of X which can be "reached from $\mathfrak{A}$" by performing at most α operations, where each operation consists of forming a denumerable union and a complement. If $\mathfrak{A}$ generates the structure of X, then the union (over α) of all the $Q_\alpha(\mathfrak{A})$ (or of the $P_\alpha(\mathfrak{A})$) is the set of all measurable subsets of X. If $\mathfrak{B}$ is a class of measurable subsets of Y and $\alpha \geqq 0$ is a denumerable ordinal number, then we define $L_\alpha(\mathfrak{A}, \mathfrak{B})$ to be the set of all functions $f: X \to Y$ such that for all $B \in Q_1(\mathfrak{B})$, $f^{-1}(B) \in Q_{\alpha+1}(\mathfrak{A})$. If X and Y are separable and $\mathfrak{A}$ and $\mathfrak{B}$ are denumerable generating sets for their respective structures, then the union (over α) of all the $L_\alpha(\mathfrak{A}, \mathfrak{B})$ is the set of all measurable transformations from X into Y. It will be denoted Y^X. In this case $L_\alpha(\mathfrak{A}, \mathfrak{B})$ is called the *Banach class*[5] of order α for $(\mathfrak{A}, \mathfrak{B})$. A subset F of Y^X is said to be of *bounded* Banach class if there is an α and denumerable generating sets $\mathfrak{A}$, $\mathfrak{B}$ such that $F \subset L_\alpha(\mathfrak{A}, \mathfrak{B})$. It is important to note that the definition of bounded Banach class is independent of the choice of $\mathfrak{A}$ and $\mathfrak{B}$, i.e. that if $F \subset L_\alpha(\mathfrak{A}, \mathfrak{B})$, then for any other generating pair $\mathfrak{A}'$, $\mathfrak{B}'$, there is an α' such that $F \subset L_{\alpha'}(\mathfrak{A}', \mathfrak{B}')$. If X and Y are separable metric spaces and Y is pathwise connected, then the Banach classes coincide with the Baire classes (for appropriate choice of $\mathfrak{A}$ and $\mathfrak{B}$).

THEOREM D. *If X and Y are separable, then F is admissible if and only if it is of bounded Banach class.*

THEOREM E. *If X and Y are separable, then every admissible subset of Y^X has a separable admissible structure.*

A space Z and its structure are called *regular* if for all $x, y \in Z$, there is a measurable set in Z containing x but not y. It is known (cf. [2]) that a space is separable and regular if and only if it is isomorphic[6] to a subspace of I, where I denotes the unit interval $[0, 1]$ with the usual Borel structure.

THEOREM F. *If X and Y are separable and regular, then every admissible subset of Y^X has a separable and regular admissible structure.*

[5] Because of the work that Banach [1] did in characterizing these classes.

[6] Two spaces are said to be *isomorphic* if there is a $1-1$ correspondence between them that preserves measurability (in both directions).

The *natural* admissible structure on a given admissible set F is defined to be the smallest admissible structure on F, if it exists. Alternatively, it may be defined to be the intersection of all the admissible structures on F, in case this is admissible. Not every admissible set need have a natural admissible structure; the counterexample is due to P. R. Halmos.

If $a \in X$ and $B \subset Y$, define $F(a, B) = \{f : f \in F, f(a) \in B\}$. It is not hard to prove that if B is measurable and a is arbitrary, then every admissible structure on F must contain $F(a, B)$. A "converse" would be that the structure generated by the $F(a, B)$ is admissible, and it would follow that it is also natural.

THEOREM G. *If X and Y are separable metric spaces and F contains continuous functions only, then F has a natural admissible structure, which is generated by the set of all $F(a, B)$, where B is measurable and a is arbitrary.*

We now give some applications. A space is said to have the *discrete* structure if every subset is measurable. Let J be the space consisting of 0 and 1 only, and K the space of all positive integers, both with the discrete structure. If X is an arbitrary space, then X^J and X^K are both admissible, and possess natural admissible structures which make them isomorphic to $X \times X$ and $\times_{i=1}^{\infty} X_i$ respectively, where the X_i are copies of X. In particular, J^K is admissible and has a natural admissible structure which makes it isomorphic to I. These results are relatively trivial or at least easily derivable from known results.

The situation changes when we pass to exponent spaces with nondiscrete structures. For example, J^I may be considered the set of all measurable subsets of I. It is not itself admissible. The set of all open subsets of I is admissible, as is the set of all closed subsets, the set of all G_δ, etc. In general, a subset F of J^I is admissible if and only if all members of F can be constructed from the open subsets of I by taking denumerable unions and intersections at most α times, where α is an arbitrary denumerable ordinal number (which is fixed for given F, but may differ for different F). I do not know whether or not every admissible subset of J^I has a natural admissible structure, but if F is admissible, then we may endow it with an admissible structure in such a way so that it will be isomorphic to a subset of I.

I^I is not admissible. The set of all continuous functions from I into I is admissible; more generally, a necessary and sufficient condition that a subset F of I^I be admissible is that there exist a denumerable ordinal number α such that all members of F are of Baire class α

at most. The set H of all continuous functions from I into I has a natural admissible structure; it is the Borel structure of H when considered as a metric space (in the uniform convergence topology). Again, I do not know whether or not every admissible subset of I^I has a natural admissible structure, but if F is admissible, we may endow it with an admissible structure in such a way so that it will be isomorphic to a subset of I.

The above theory may be applied to give a generalization of Kuhn's theorem [3] about optimal behavior strategies in games of perfect recall, to games in which there may be a continuum of alternatives at some of the moves.

A fuller account of the theory outlined above, together with proofs, will be published elsewhere.[7]

References

1. S. Banach, *Über analytisch darstellbare Operationen in abstrakten Räumen*, Fund. Math. vol. 17 (1931) pp. 283–295.

2. G. W. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc. vol. 85 (1957) pp. 134–165.

3. H. W. Kuhn, *Extensive games and the problem of information*, Contributions to the Theory of Games II, Princeton University Press, 1953, pp. 245–266.

[7] R. J. Aumann, *Borel Structures for Function Spaces*, Illinois J. Math. vol. 5 (1961) pp. 614–630 [Chapter 66].