

1 Introduction

This paper grew out of an attempt to translate into mathematical formulas what people may argue when faced with a cooperative n -person game described by a characteristic function.

The basic difficulty in n -person game theory is due to the lack of a clear meaning as to what is the purpose of the game. Certainly, the purpose is not just to get the maximum amount of profits, because if every player will demand the maximum he can get in a coalition, no agreement will be reached. Thus, one decides that the purpose of the game is to reach some kind of stability, to which the players would or should agree if they want any agreement to be enforced. This stability should reflect in some sense the power of each player, which results from the rules of the game.

In this paper, we assume that all the players can “bargain” together, with perfect communication, and settle at a “stable” outcome which is based on the “threats” and “counter threats” that they possess. The set of all the stable outcomes, called the *bargaining set*,¹ is defined in Section 2 and some of its properties are discussed. In particular, this set can always be determined by solving systems of algebraic linear inequalities.

The bargaining sets for the 2- and 3-person games are fully described (Sections 3, 4, 5) and some cases of 4-person games are treated, in which not all the coalitions are permissible (Sections 6, 8).

Some counterexamples for various conjectures, as well as existence theorems, are treated in Section 7, and possible modifications are suggested in Sections 9 and 10. In Section 11 we discuss some similarities and deviations between our theory and other solution concepts, such as W. Vickrey’s concept of self-policing patterns [8] and D. Luce’s concept of ψ -stability [3]. We also outline how our theory can be modified if the game is given by Thrall’s characteristic function [7] or by the Aumann-Peleg characteristic function for cooperative games without side payments.

We conclude by pointing out that “central parts” of the von Neumann-Morgenstern solution for some games appear also in the bargaining set. The reason for this phenomenon is obscure to us.

This chapter originally appeared in *Advances in Game Theory*, Annals of Mathematics Studies 52, edited by M. Dresher, L. S. Shapley, and A. W. Tucker, pp. 443–476, Princeton University Press, Princeton, 1964. Reprinted with permission.

This research was sponsored in part by the Office of Naval Research under Nonr-1858(16), and in part by the Carnegie Corporation of New York.

1. The definition of the bargaining set appeared in [1].

2 The Bargaining Set

We shall consider an n -person cooperative game Γ , described by its characteristic function. More precisely, a set $N = \{1, 2, \dots, n\}$ of n players is given, together with a collection $\{B\}$ of non-empty subsets B of N , called *permissible coalitions*. For each $B, B \in \{B\}$, a number $v(B)$ is given and it is called *the value of the coalition B* .

For the sake of simplicity, we shall assume throughout this paper that all 1-person coalitions are in $\{B\}$ and have a zero value, i.e.,

$$i \in \{B\}, \quad v(i) = 0. \quad (2.1.a)$$

In addition, we shall also assume that

$$v(B) \geq 0, \quad B \in \{B\}. \quad (2.1.b)$$

It will turn out later that no essential change will occur if we add to $\{B\}$ all other, non-permissible coalitions, assigning them the value zero.

A *payoff configuration* (p.c.) will now be defined as an expression of the form

$$(x; \mathcal{B}) \equiv (x_1, x_2, \dots, x_n; B_1, B_2, \dots, B_m), \quad (2.2)$$

where B_1, B_2, \dots, B_m are mutually disjoint sets of $\{B\}$ whose union is N , i.e.,

$$B_j \cap B_k = \phi, \quad j \neq k; \quad \bigcup_{j=1}^m B_j = N; \quad (2.3)$$

and the x_i are real numbers which satisfy

$$\sum_{i \in B_j} x_i = v(B_j); \quad j = 1, 2, \dots, m. \quad (2.4)$$

A p.c. is therefore a representation of a possible outcome of the game, in which the players divide themselves into the coalitions B_1, B_2, \dots, B_m , each coalition shares its value among its members, and each player receives the amount $x_i, i = 1, 2, \dots, n$.

When people are faced with such a game, each one trying to get as high an amount as he thinks he can get, it is reasonable to expect that some of the p.c.'s will never form. E.g., one does not expect that a p.c. will occur with $x_{i_0} < 0$, since the player i_0 alone can secure more by playing as a 1-person coalition. We are willing to make a strong assumption, namely, that the outcome (2.2) will be a *coalitionally rational p.c.* (c.r.p.c.),² i.e., for each $B, B \in \{B\}, B \subset B_j, j = 1, 2, \dots, m$,

2. In R. J. Aumann and M. Maschler [1], a c.r.p.c. is called a p.c.

$$\sum_{i \in B} x_i \geq v(B). \quad (2.5)$$

Thus, we assume that a coalition will not form if some of its members can obtain more by themselves forming a permissible coalition.

The assumption of coalitional rationality differs from the assumption of belonging to the core by the restricting condition $B \subset B_j$. This restriction avoids some of the difficulties which arise when dealing with games whose core is empty. (See R. D. Luce and H. Raiffa [3].)

In itself, the coalitional rationality assumption is a very strong one, as it forces the game to be essentially superadditive within those coalitions which are actually formed. Indeed, a coalition whose value is less than the sum of the values of disjoint subcoalitions cannot occur in any c.r.p.c., and can as well be declared non-permissible or have its value be replaced by zero. Moreover, this assumption is open to the same theoretical objections which are discussed at length in R. D. Luce and H. Raiffa [3]. As a matter of fact, our theory can be developed without the coalitional rationality assumption, as indicated in Section 10. Nevertheless, as we are only interested in "stable" outcomes, we feel it instructive to make this assumption.

Several phenomena can be observed when watching people who are confronted with a game such as described above. Usually, negotiations start, each one tries to get at least as much as he expects, and at the same time there is an attempt to enter into a "safe" coalition. This latter factor applies, in particular, to those coalitions which are planned to operate for a long period. The search for "safety" gives rise to feelings of sympathy and antipathy which play an important role in the final decisions. Guarantees of all kinds are demanded, contracts are signed, etc. *If people do not feel safe enough, they often do not enter a coalition even if they can win more in it.*

The demand for safety is usually considered legitimate and a sound way to convince the partners to get a smaller amount of profit *in order that no one in the coalition will feel deprived*. There is a desire for "fair play," which can be achieved in various ways. Often it is accepted that "if all things are equal" it is "fair" to divide the profits equally. Sometimes people share the profits according to some fixed ratio established by other precedents, etc.

If "all things" are *not* equal, people will still be happy with their coalition if they agree that the "stronger" partners will get more. Thus, during the negotiations prior to the coalition formation, each player will try to convince his partners that in some sense he is strong. This he can try in various ways, among which an important factor is his ability to show that

he has other, perhaps better, alternatives. His partners, besides pointing out their own alternatives, may argue in return that even without his help they can perhaps keep their proposed shares. Thus, a negotiation quite often takes the form of a sequence of "threats" and "counter threats," or "objections" against "counter objections." It is this principle that we shall try to formulate mathematically. It seems that a certain kind of stability is reached if all objections can be answered by counter objections.

Perhaps it is not enough that any objection by one person could be met. It is possible that a subset of the players of a coalition unite, during the negotiation period, and threaten another subset. If we insist on a strong stability, we have to take care also of such threats. This, in fact, will be done.

To be sure, there are other means used during the bargaining period, such as threats based on the so-called "interpersonal comparison of utilities," sanctions in other games, propaganda, etc. These will be ignored in this paper.

The following example will illustrate our purpose. Let $n = 3$, $v(1) = v(2) = v(3) = v(123) = 0$, $v(12) = 100$, $v(13) = 100$, $v(23) = 50$. Consider the p.c.

$$(80, 20, 0; 12, 3). \quad (2.6)$$

Now, player 2 can object by pointing out that in the p.c.

$$(0, 21, 29; 1, 23) \quad (2.7)$$

he and player 3 get more. Player 1 has no counter objection because he cannot keep his 80 while offering player 3 at least 29. Thus, (2.6) is unstable. On the other hand,

$$(75, 25, 0; 12, 3) \quad (2.8)$$

is stable. An objection of player 2, e.g.,

$$(0, 26, 24; 1, 23), \quad (2.9)$$

can be met by a counter objection

$$(75, 0, 25; 13, 2); \quad (2.10)$$

or an objection of player 1, e.g.,

$$(76, 0, 24; 13, 2) \quad (2.11)$$

can be met by the counter objection

$$(0, 25, 25; 1, 23). \quad (2.12)$$

In these counter objections, the threatened player can keep his share and offer his partners at least what the player who objects offered. It will turn out that the only stable p.c.'s in this game are

$$(0, 0, 0; 1, 2, 3); \quad (75, 25, 0; 12, 3); \quad (75, 0, 25; 13, 2); \quad (0, 25, 25; 1, 23). \quad (2.13)$$

Let it be said at once that our paper was largely motivated by the fact that most of our friends, to whom this game was presented, started their considerations from the p.c.'s (2.8) and (2.10). We tried to find what characterizes these p.c.'s and how they can be generalized to more complicated cases.

Let Γ be a game, as described above. Let K be a non-empty subset of the set of players N . A player i will be called a *partner of K in a p.c.* $(x; \mathcal{B})$, if he is a member of a coalition in \mathcal{B} which intersects K . The set $P[K; (x; \mathcal{B})]$ of all the partners of K in (x, \mathcal{B}) is, therefore,

$$P[K; (x; \mathcal{B})] \equiv \{i \mid i \in B_j, B_j \cap K \neq \phi\}. \quad (2.14)$$

Note that $K \subset P[K; (x; \mathcal{B})]$; i.e., each member of K is also a partner of K , contrary to everyday usage. K needs only the consent of its partners in order to get its part of x .

DEFINITION 2.1 Let $(x; \mathcal{B})$ be a coalitionally rational payoff configuration (2.2), (2.5) for a game Γ . Let K and L be non-empty disjoint subsets of a coalition B_j which appears in $(x; \mathcal{B})$. An *objection* of K against L in (x, \mathcal{B}) will be a c.r.p.c.

$$(y; \mathcal{C}) \equiv (y_1, y_2, \dots, y_n; C_1, C_2, \dots, C_l) \quad (2.15)$$

for which

$$P[K; (y; \mathcal{C})] \cap L = \phi, \quad (2.16)$$

$$y_i > x_i \quad \text{for all } i, i \in K, \quad (2.17)$$

$$y_i \geq x_i \quad \text{for all } i, i \in P[K; (y; \mathcal{C})]. \quad (2.18)$$

Verbally, in their objection, players K claim that, without the aid of players L ((2.16)), they can get more in another c.r.p.c. ((2.17)), and the new situation is reasonable because their new partners do not get less than what they got in the previous p.c. ((2.18)).

DEFINITION 2.2 Let (x, \mathcal{B}) be a coalitionally rational payoff configuration (2.2), (2.5) in a game Γ , and let $(y; \mathcal{C})$ be an objection of a set K against a set L in $(x; \mathcal{B})$, $K, L \subset B_j$. A *counter objection* of L against K is a c.r.p.c.

$$(z; \mathcal{D}) \equiv (z_1, z_2, \dots, z_n; D_1, D_2, \dots, D_k) \quad (2.19)$$

for which

$$P[L; (z; \mathcal{D})] \not\subseteq K, \quad (2.20)$$

$$z_i \geq x_i \quad \text{for all } i, i \in P[L; (z; \mathcal{D})], \quad (2.21)$$

$$z_i \geq y_i \quad \text{for all } i, i \in P[L; (z; \mathcal{D})] \cap P[K; (y; \mathcal{C})]. \quad (2.22)$$

Verbally, in their counter objection, players L claim that they can hold their original properties ((2.21)), promise their partners at least their original share ((2.21)), and if they need partners of K in his objection, they can give them not less than what they were offered in the objection ((2.22)). Sometimes, the members of L have to use the tactics of “divide and rule” by using members of K as partners, but they may not use *all* members of K ((2.20)).

DEFINITION 2.3 A c.r.p.c (x, \mathcal{B}) is called stable if for each objection of a K against an L in (x, \mathcal{B}) there is a counter objection of L against K . The *bargaining set* \mathfrak{M} of a game Γ is the set of all stable c.r.p.c.’s.

The feeling of “safety” suggested by this definition lies in the assurance that all threats *within a coalition* can be met. It may be felt, perhaps, that there is a lack of symmetry when comparing (2.16) and (2.20), but the situation is not symmetric in the first place. An objection (2.15) can serve, in general, as an objection of K (or another “ K ”) against various groups “ L ,” each one of which has to have a counter objection.

To be sure, even if there is a desire for a stability as demanded in the p.c.’s of the bargaining set, this does not mean that the outcome will belong to \mathfrak{M} . A player, e.g., may agree to sacrifice some of his profits in order to make sure that he enters a coalition. Other factors, mentioned above, may also cause deviations from \mathfrak{M} . However, if the demand for stability is strong enough, we hope that the outcome will not be too far from \mathfrak{M} ; in this sense the theory has a normative aspect. Moreover, as the number of the players increases, there arise many possible threats, and, using the concepts involved in the definition, one may compute and show the players where they are “safe” and what threats they do possess. This is another normative aspect.³

The bargaining set is never empty. Indeed, $(0, 0, \dots, 0; 1, 2, \dots, n)$ always belongs to \mathfrak{M} .

3. Results close to the bargaining set have been observed in an experimental study [4].

In a coalition of zero value, any objection (if there is one) can be countered by the other players playing as 1-person coalitions.

A dummy always gets zero in a c.r.p.c., therefore he cannot belong to any objecting K . On the other hand, he can always keep his zero by playing alone. He can be of no use for any objection or counter objection, since the same can be effected without his help. Thus, a dummy has no essential effect on \mathfrak{M} .

The definition of \mathfrak{M} does not use "interpersonal comparisons of utilities" and it is independent of the names of the players.

THEOREM 2.1 The bargaining set \mathfrak{M} of a game Γ can be represented as the set of solutions of a conjunctive-disjunctive⁴ system of linear inequalities involving the x_i as unknowns. It is, therefore, a union of a finite number of polyhedral convex sets in the n -space with coordinates (x_1, x_2, \dots, x_n) .

*Proof.*⁵ In any finite expression with coordinates which has the form of quantifiers followed by linear inequalities connected by the words "or" and "and," the free variables—if such exist—which satisfy the expression are those and only those which satisfy a certain disjunctive-conjunctive system of linear inequalities. This is a known theorem in logic, but for the sake of completeness we sketch the proof. It is sufficient to prove the theorem when there is only one quantifier. Moreover, we may assume that this quantifier is \exists , because $\forall = \sim \exists \sim$. The theorem now follows from the fact that the projection of a polyhedron is a polyhedron.

3 The Two-Person Game

The bargaining set \mathfrak{M} for the game:

$$v(1) = v(2) = 0 \quad v(12) = a \geq 0, \quad (3.1)$$

consists of all possible c.r.p.c.'s; i.e.,

$$\begin{cases} (0, 0; 1, 2) \\ (x_1, x_2; 12) \quad x_1 + x_2 = a, \quad x_1 \geq 0, \quad x_2 \geq 0. \end{cases} \quad (3.2)$$

Indeed, there are no possible objections.

4. I.e., a system of linear inequalities connected by the words "or" and "and".

5. We are indebted to Professor M. Rabin and Professor A. Robinson for pointing this out.

4 The 3-Person Game. Permissible Coalitions of Less Than Three Players

In this section we shall study the game:

$$v(1) = v(2) = v(3) = 0; \quad v(12) = a; \quad v(23) = b; \quad v(13) = c; \quad a, b, c \geq 0. \quad (4.1)$$

THEOREM 4.1 In the game (4.1), essentially two cases arise.

Case A. If a, b, c satisfy the "triangle inequality"

$$a \leq b + c, \quad b \leq a + c, \quad c \leq a + b, \quad (4.2)$$

then the bargaining set \mathfrak{M} is:

$$\left\{ \begin{array}{l} (0, 0, 0; 1, 2, 3) \\ \left(\frac{a+c-b}{2}, \frac{a+b-c}{2}, 0; 12, 3 \right) \\ \left(\frac{a+c-b}{2}, 0, \frac{c+b-a}{2}; 13, 2 \right) \\ \left(0, \frac{a+b-c}{2}, \frac{c+b-a}{2}; 1, 23 \right) \end{array} \right. \quad (4.3)$$

Case B. If, e.g., $a > b + c$, then the bargaining set \mathfrak{M} is:

$$\left\{ \begin{array}{l} (0, 0, 0; 1, 2, 3) \\ (c \leq x_1 \leq a-b, a-x_1, 0; 12, 3) \\ (c, 0, 0; 13, 2) \\ (0, b, 0; 1, 23) \end{array} \right. \quad (4.4)$$

Before proving this theorem, we shall give some illustrations which will throw some light on the nature of the bargaining sets.

Example 1 Let $a = 100, b = 100, c = 50$. The triangle inequality is satisfied, and therefore \mathfrak{M} is discrete: $\{(0, 0, 0; 1, 2, 3), (25, 75, 0; 12, 3), (25, 0, 25; 13, 2), (0, 75, 25; 1, 23)\}$.

One can approach this solution also by the following intuitive argument: Suppose that player 1 receives α , then player 2 gets $100 - \alpha$ and he is thus willing to pay player 3 at most $100 - (100 - \alpha) = \alpha$. Thus, player 3 will be willing to pay player 1 at most $50 - \alpha$. If $50 - \alpha > \alpha$, then player 1 will prefer to join player 3. This will cause player 2 to agree to get less. If $50 - \alpha < \alpha$, player 2 will demand more as he will get more from player

3 if player 1 insists on getting α . Thus an equilibrium will be reached only if $\alpha = 50 - \alpha$, in which case $\alpha = 25$.

Example 2 The above argument fails in the case $a = 20$, $b = 30$, $c = 100$. Here one obtains $\alpha = 45$ in which case player 2 will lose money. This he can avoid by playing alone. Our bargaining set is no longer discrete: $\{(0, 0, 0; 1, 2, 3) (20 \leq x_1 \leq 70, 0, 100 - x_1; 13, 2) (20, 0, 0; 12, 3) (0, 0, 30; 1, 23)\}$. One can reason as follows: Player 1, being in the coalition 13, will not be satisfied in getting less than 20, since otherwise he will do better by joining player 2. Similarly, player 3 will demand at least 30. Fortunately, both demands can be satisfied, and player 2 cannot cause any harm since he is a weak player.

Example 3 Let $a = 100$, $b = 100$, $c = 0$. We observe that the bargaining set is again discrete: $\{(0, 0, 0; 1, 2, 3), (0, 100, 0; 12, 3), (0, 0, 0; 13, 2), (0, 100, 0; 1, 23)\}$. This solution reflects the character of an “unrestricted competition” in our bargaining set. Indeed, player 2 can practically receive the amount 100 because whatever the positive demand of player 1 will be, player 3 will be “satisfied” in getting less, and vice versa. One observes that our theory does not take into account the psychological threat that player 2 may also “lose” his profit 100, and probably will therefore be willing to pay some amount in order to be in a coalition with player 1 or with player 3. In practical situations several side conditions may come into consideration such as: (1) It may be *customary* not to enter a coalition unless a certain minimum amount or percentage of profit is guaranteed in advance. (2) A “cartel” agreement is decided between player 1 and player 3, in which both of them declare not to enter a coalition with player 2 without getting at least a certain amount of profit. (3) There is enforced a “cartel” or an “anti-cartel” law in the country. (4) It is known that in order to ensure a certain profit, one is willing to give up a certain amount or percentage in order to “push” an equilibrium situation to one’s side.

Proof of Theorem 4.1 Certainly, $(0, 0, 0; 1, 2, 3) \in \mathfrak{M}$.

Next, let us examine under what circumstances a payoff configuration $(x_1, x_2, 0; 12, 3)$ can belong to the bargaining set. It should be coalitionally rational, and therefore it must satisfy

$$x_1 \geq 0, \quad x_2 \geq 0; \quad x_1 + x_2 = v(12). \quad (4.5)$$

LEMMA 1 A necessary and sufficient condition that player 1 has no objection is:

$$x_1 \geq v(13). \quad (4.6)$$

Proof Indeed, if $x_1 \geq v(13)$, then player 1 has no objection either by playing alone (see (4.5)) or by participating in the coalition 13. If $x_1 < v(13)$, then player 1 can suggest the objection

$$\left(\frac{v(13) + x_1}{2}, 0, \frac{v(13) - x_1}{2}; 13, 2 \right). \quad (4.7)$$

This is a coalitionally rational payoff configuration.

LEMMA 2 A necessary and sufficient condition that player 1 has an objection and to each such objection player 2 has a counter objection, is:

$$x_1 < v(13), \quad (4.8)$$

$$x_1 - x_2 \geq v(13) - v(23) \quad \text{or} \quad x_2 = 0. \quad (4.9)$$

Proof Indeed, if (4.8) and (4.9) hold, then, by Lemma 1, player 1 has an objection. This can only be (see (4.5)) of the form

$$(x_1 + \varepsilon, 0, v(13) - x_1 - \varepsilon; 13, 2), \quad (4.10)$$

where ε is a sufficiently small positive number. If $x_2 = 0$, then (4.10) is itself also a counter objection; otherwise,

$$(0, v(23) - v(13) + x_1 + \varepsilon, v(13) - x_1 - \varepsilon; 1, 23) \quad (4.11)$$

is a possible counter objection. By (4.8), player 2 will now receive even more than x_2 . If (4.8) does not hold, then there is no objection for player 1, by Lemma 1. If (4.8) holds, but

$$x_2 > 0 \quad \text{and} \quad x_1 - x_2 < v(13) - v(23), \quad (4.12)$$

then player 1 can object by (4.10), choosing ε so small that $v(23) - v(13) + x_1 + \varepsilon < x_2$. Now, player 2 does not have any counter objection, either by playing alone or by forming a coalition with player 3.

Summing up, and making the necessary permutations, we obtain:

LEMMA 3 A necessary and sufficient condition that a payoff configuration $(x_1, x_2, 0; 12, 3)$ will belong to the bargaining set \mathfrak{M} , is that x_1 and x_2 will satisfy (4.5) as well as at least one of the following columns:

$$x_1 \geq v(13) \quad \left| \quad \begin{array}{l} x_1 < v(13) \\ x_2 = 0 \end{array} \quad \left| \quad \begin{array}{l} x_1 < v(13) \\ x_1 - x_2 \geq v(13) - v(23) \end{array} \right. \quad (4.13)$$

and also at least one of the following columns:

$$x_2 \geq v(23) \quad \left| \quad \begin{array}{l} x_2 < v(23) \\ x_1 = 0 \end{array} \quad \left| \quad \begin{array}{l} x_2 < v(23) \\ x_2 - x_1 \geq v(23) - v(13) \end{array} \right. \quad (4.14)$$

Taking into account that $x_1 + x_2 = a$, these inequalities reduce to

$$\begin{array}{l|l|l} 0 \leq x_1 \leq a & x_1 < c & 0 \leq x_1 \leq a \\ c \leq x_1 & x_1 = a & \frac{a+c-b}{2} \leq x_1 < c \end{array} \quad (4.15)$$

$$\begin{array}{l|l|l} x_1 \leq a-b & a-b < x_1 & a-b < x_1 \leq \frac{a+c-b}{2} \\ & x_1 = 0 & \end{array} \quad (4.16)$$

We now use the assumption $a, b, c \geq 0$, and the inequalities (4.15), (4.16). A detailed calculation yields the following results:

Case A. If a, b, c satisfy the “triangle inequalities” (4.2), then

$$x_1 = \frac{a+c-b}{2} \quad (4.17)$$

is the only solution.

Case B. If $a > b + c$, then each x_1 satisfying

$$c \leq x_1 \leq a - b, \quad (4.18)$$

is a solution, and there are no other solutions.

Case C. If $b > a + c$, then $x_1 = 0$ is the only solution.

Case D. If $c > a + b$, then $x_1 = a$ is the only solution.

These are the only possible cases, they exclude each other, and therefore the proof of Theorem 4.1 has been completed.

5 The General 3-Person Game

Let us add the coalition 123 with its value $v(123) = d \geq 0$ to the game treated in Section 4. This coalition will have no effect on the previous p.c.’s of the bargaining set. Indeed, this coalition cannot be used for objections and counter objections, because it contains all the players L and K . Thus, it only remains to find out under what condition does a p.c. $(x_1, x_2, x_3; 123)$ belong to the new bargaining set.

As it should be coalitionally rational, it is necessary that x_1, x_2, x_3 satisfy:

$$\begin{aligned} x_1, x_2, x_3 \geq 0; \quad x_1 + x_2 \geq a, \quad x_2 + x_3 \geq b, \quad x_1 + x_3 \geq c; \\ x_1 + x_2 + x_3 = d. \end{aligned} \quad (5.1)$$

On the other hand, if (5.1) is satisfied, there can be no objection and hence this pair belongs to \mathfrak{M} .

In order that the inequalities (5.1) have at least one solution, it is necessary and sufficient that

$$d \geq a, b, c, \quad d \geq \frac{a+b+c}{2}. \quad (5.2)$$

We have thus proved:

THEOREM 5.1 In the 3-person game for which

$$\begin{aligned} v(1) = v(2) = v(3) = 0, \quad v(12) = a, \quad v(23) = b, \\ v(13) = c, \quad v(123) = d, \quad a, b, c, d \geq 0, \end{aligned}$$

the bargaining set \mathfrak{M} consists of the p.c.'s given by Theorem 4.1, and also of the p.c.'s $(x_1, x_2, x_3; 123)$ which satisfy (5.1). The latter p.c.'s exist if and only if (5.2) is satisfied.

6 The 4-Person Game. Coalitions of 1 Person and 3 Persons

Consider the four-person game, in which the permissible coalitions are all the single-person and the three-person coalitions. Let their values be

$$\begin{cases} v(1) = v(2) = v(3) = v(4) = 0, & v(123) = a, & v(124) = b, \\ v(134) = c, & v(234) = d, & a, b, c, d \geq 0. \end{cases} \quad (6.1)$$

Evidently $(0, 0, 0, 0; 1, 2, 3, 4)$ belongs to the bargaining set \mathfrak{M} . Similar considerations to those which were used in Section 4 lead to the inequalities which are listed in Appendix 1. These inequalities express a necessary and sufficient condition in order that the payoff configuration $(x_1, x_2, x_3; 123, 4)$ belongs to the bargaining set.

We omit the calculations, which are somewhat lengthy but easy, and state the results. There are essentially four different cases:

Case A. If

$$\begin{cases} 2a \leq b + c + d, \\ 2b \leq a + c + d, \\ 2c \leq a + b + d, \\ 2d \leq a + b + c, \end{cases} \quad (6.2)$$

then the bargaining set is

$$\left\{ \begin{array}{l} (0 \quad 0 \quad 0 \quad 0 \quad ; \quad 1,2,3,4) \\ \left(\frac{a+b+c-2d}{3} \quad \frac{a+b+d-2c}{3} \quad \frac{a+c+d-2b}{3} \quad 0 \quad ; \quad 123,4 \right) \\ \left(\frac{a+b+c-2d}{3} \quad \frac{a+b+d-2c}{3} \quad 0 \quad \frac{b+c+d-2a}{3} \quad ; \quad 124,3 \right) \\ \left(\frac{a+b+c-2d}{3} \quad 0 \quad \frac{a+c+d-2b}{3} \quad \frac{b+c+d-2a}{3} \quad ; \quad 134,2 \right) \\ (0 \quad \frac{a+b+d-2c}{3} \quad \frac{a+c+d-2b}{3} \quad \frac{b+c+d-2a}{3} \quad ; \quad 234,1) \end{array} \right. \quad (6.3)$$

Case B. If

$$\left\{ \begin{array}{l} 2a > b + c + d, \\ 2b \leq a + c + d, \quad b \leq c + d, \\ 2c \leq a + b + d, \quad c \leq b + d, \\ 2d \leq a + b + c, \quad d \leq b + c, \end{array} \right. \quad (6.4)$$

then the bargaining set is:

$$\left\{ \begin{array}{l} (0 \quad 0 \quad 0 \quad 0; \quad 1,2,3,4) \\ (x_1 \quad a - x_1 - x_3 \quad x_3 \quad 0; \quad 123,4) \\ \left(\frac{b+c-d}{2} \quad \frac{b+d-c}{2} \quad 0 \quad 0; \quad 124,3 \right) \\ \left(\frac{b+c-d}{2} \quad 0 \quad \frac{c+d-b}{2} \quad 0; \quad 134,2 \right) \\ (0 \quad \frac{b+d-c}{2} \quad \frac{c+d-b}{2} \quad 0; \quad 234,1) \end{array} \right. \quad (6.5)$$

Here, x_1 and x_3 satisfy the inequalities

$$0 \leq x_1 \leq a - d, \quad 0 \leq x_3 \leq a - b, \quad c \leq x_1 + x_3 \leq a. \quad (6.6)$$

Case C. If

$$\left\{ \begin{array}{l} 2a > b + c + d, \\ 2b \leq a + c + d, \quad b > c + d, \\ 2c \leq a + b + d, \\ 2d \leq a + b + c, \end{array} \right. \quad (6.7)$$

then the bargaining set is:

$$\left\{ \begin{array}{l} (0 \quad 0 \quad 0 \quad 0; 1,2,3,4) \\ (x_1 \quad a - x_1 - x_3 \quad x_3 \quad 0; 123,4) \\ (c \leq \xi_1 \leq b - d \quad b - \xi_1 \quad 0 \quad 0; 124,3) \\ (c \quad 0 \quad 0 \quad 0; 134,2) \\ (0 \quad d \quad 0 \quad 0; 234,1) \end{array} \right. \quad (6.8)$$

Here, x_1 and x_3 satisfy the inequalities (6.6).

Case D. If

$$\left\{ \begin{array}{l} 2a > b + c + d, \\ 2b > a + c + d, \\ a \geq b, \end{array} \right. \quad (6.9)$$

the bargaining set is the same as in Case C.

Only Case *A* is completely discrete; all other cases contain the continuum (6.6). Equations (6.2) can be considered as a generalization of the triangle inequalities. In fact, it follows from (6.2) that any three of the numbers a, b, c, d satisfy the triangle inequalities. Moreover, an equality $a = b + c$, for example, can occur only if $a = d$. The converse does not hold. (E.g., $a = 8, b = c = d = 5$.)

It is possible to approach the bargaining set in Case *A* as follows: If players 1 and 2 get α and β , respectively, then player 3 gets $a - \alpha - \beta$ in the coalition 123. With these values, player 4 will get $b - \alpha - \beta$ in the coalition 124, $c - \alpha + \beta$ in the coalition 134, and $d - a + \alpha$ in the coalition 234. In order that no coalition can exert threats on others, it is necessary and sufficient that

$$b - \alpha - \beta = c - \alpha + \beta = d - a + \alpha. \quad (6.10)$$

Hence

$$\alpha = \frac{a + b + c - 2d}{3}, \quad \beta = \frac{a + b + d - 2c}{3}. \quad (6.11)$$

If

$$2a \geq b + c + d, \quad a \geq b, c, d, \quad (6.12)$$

then the coalition 123 is strong and player 4 cannot get more than zero. If we decide to omit him from the game, and look at each of the remaining two persons in a coalition which contained him as a new 2-person coalition which has the same value as before, we get a 3-person game in which $v(123) = a, v(12) = b, v(23) = d, v(13) = c, v(1) = v(2) = v(3) = 0$.

A comparison with the previous two sections shows that the bargaining set of the new game is essentially the same as the game treated in Cases *B*, *C*, *D*, and Case *A*, if equality holds in the first relation of (6.12).

(Note that each of the systems (6.4) and (6.7) implies $a > b, c, d$, (6.9) implies $a \geq b, a > c, d$, and (6.7) as well as (6.9) implies $b > c + d$.)

We can therefore conclude:

THEOREM 6.1 A 4-person game, in which the permissible coalitions are all the 1-person and 3-person coalitions always has a bargaining set in which all possible partitions into coalitions appear. The set is discrete if and only if (6.2) is satisfied. If (6.12) holds, the bargaining set is essentially the same as the one of a full 3-person game obtained from the original one by deleting a player who belongs to the complement of a maximal valued coalition. This player always gets 0.

Remark 1 The same situation occurs in a 3-person game in which the only non-permissible coalition is the 3-person coalition. If the triangle inequality does not hold, then one coalition is strong enough to reduce the game to a 2-person game with essentially the same bargaining set. The weak player gets 0.

Remark 2 The conditions (6.12) are necessary and sufficient for the existence of a c.r.p.c. $(x_1, x_2, x_3, 0; 123, 4)$ such that

$$x_1, x_2, x_3 \geq 0, \quad x_1 + x_2 \geq b, \quad x_1 + x_3 \geq c, \quad x_2 + x_3 \geq d. \quad (6.13)$$

Obviously, such p.c.'s cannot be objected against. However, in any c.r.p.c, in which player 4 is in a 3-person coalition and receives more than 0, there exists an objection against player 4 which cannot be countered. Thus, the coalition 123 "dictates" everything; this is why player 4 cannot claim more than 0.

Remark 3 The following *ad hoc* rule serves for the discrete case: The value of each coalition is equally divided among its members. If a person enters a coalition he gets the sum of "his shares" minus the sum of the "shares" which his partners get from coalitions which do not include him. For example: The first player's shares are $a/3, b/3, c/3$. If he is entering the coalition 123, his partners have their shares $d/3, d/3$ from the coalition 234, which is that coalition that does not contain player 1; therefore, this player gets

$$\frac{a}{3} + \frac{b}{3} + \frac{c}{3} - \frac{d}{3} - \frac{d}{3} \quad (6.14)$$

if he enters the coalition 123.

The same rule applies also to the 3-person game, with 1- and 2-person coalitions, in the discrete case.

Remark 4 The discrete case exhibits a game with a nonnegative “3-quota.” Each player always receives his “quota” in the bargaining set if he succeeds in becoming a member of a 3-person coalition. Quota games are treated in [5], and the results of this section are further generalized in [6].

7 Existence Theorems. Counterexamples

DEFINITION 7.1 A permissible coalition in a game Γ will be called *effective* if it is possible to divide its value among its members in such a way that no permissible sub-coalition can alone make more.

Condition (5.2), for example, is a necessary and sufficient condition that the coalition 123 will be effective in the game treated in Section 5.

Clearly, we can assume that all subsets of N are permissible coalitions and that those having a positive value are effective, since we are dealing only with c.r.p.c.'s. The zero-valued coalitions will be called *trivial coalitions*.

The first question which may arise is whether each partition of the players, in which the only trivial coalitions are 1-person coalitions, is represented in \mathfrak{M} . The answer is *no*.

Example 7.1 $n = 5$, the non-trivial coalitions are 12, 35, 134, 2345, with values:

$$v(12) = 10, \quad v(35) = 85, \quad v(134) = 148, \quad v(2345) = 160. \quad (7.1)$$

Consider the coalitionally rational payoff configuration

$$(\alpha, \beta, 0, 0, 0; 12, 3, 4, 5), \quad (7.2)$$

where, of course, $0 \leq \alpha \leq 10$, $\alpha + \beta = 10$. Now, player 1 can object by

$$(11, 0, 29, 108, 0; 134, 2, 5). \quad (7.3)$$

This objection is *justified*—i.e., player 2 has no counter objection—if $\alpha < 10$. Indeed, any attempt of player 2 to keep his positive share β will end with a coalitionally non-rational p.c. Thus, (7.2) can belong to \mathfrak{M} only if $\alpha = 10$, $\beta = 0$. But this case is also ruled out, since now player 2 has a justified objection: $(0, 1, 100, 44, 15; 1, 2345)$.

Let Γ be a game, some of the values of the coalitions of which are positive. Is it possible that no p.c. belongs to \mathfrak{M} unless all the players get

zero? In other words—is it possible that, in spite of some coalitions having a positive value, it would be worthless to enter into such coalitions, if one insists on the stability demanded by the definition of \mathfrak{M} ? This, in fact, may happen as the following example shows:

*Example 7.2*⁶

$$\left\{ \begin{array}{l} v(12b) = 1, \quad b = 3, 4, 5, 6. \\ v(1ab) = 1, \quad a = 3, 4; \quad b = 5, 6. \\ v(2pq) = 1, \quad p = 3, \quad q = 4 \quad \text{or} \quad p = 5, \quad q = 6. \\ v(3456) = 1, \\ v(B) = 1, \quad B \text{ contains at least one of the} \\ \quad \quad \quad \text{above-mentioned coalitions.} \\ v(B) = 0, \quad \text{otherwise.} \end{array} \right. \quad (7.4)$$

It is a long but easy computation to verify that for this game $(x; \mathcal{B}) \in \mathfrak{M}$ implies $x_i = 0, i = 1, 2, \dots, 6$.

The following theorem might be helpful in gaining some more insight into the nature of the bargaining set \mathfrak{M} .

THEOREM 7.1 Let Γ be an n -person game, in which 12 is a permissible coalition. Let $\mathcal{B}^0 \equiv 12, B_2, \dots, B_m$ be a fixed partition. Let $(x; \mathcal{B}^0) \equiv (x_1, x_2, \dots, x_n; \mathcal{B}^0)$ be a c.r.p.c. and let J be the set of all the numbers $\sigma_1, 0 \leq \sigma_1 \leq v(12)$, such that player 1 has a justified objection⁷ against player 2, in $(\sigma_1, v(12) - \sigma_1, x_3, x_4, \dots, x_n; \mathcal{B}^0)$; then J is an open set relative to the closed interval $[0, v(12)]$.

Proof If

$$(x_1, x_2, \dots, x_n; 12, B_2, \dots, B_m) \quad (7.5)$$

is a coalitionally rational payoff configuration, then so is also

$$(x_1 + \varepsilon, x_2 - \varepsilon, x_3, \dots, x_n; 12, B_2, \dots, B_m), \quad (7.6)$$

provided that $-x_1 \leq \varepsilon \leq v(12) - x_1$.

If $x_1 \in J$, then $\delta \equiv v(12) - x_1 > 0$, since otherwise player 2 can counter object by playing alone.

Let $(y; \mathcal{C})$ be an objection of player 1 against player 2; then $y_1 > x_1$. Let z_2 be the maximum that player 2 can get by joining a coalition such

6. This game was given by J. von Neumann and O. Morgenstern [9], pp. 467–469, as an example of a simple game which is not a weighted majority game and for which no main simple solution exists.

7. By the term “a justified objection” we mean an objection which has no counter objection.

that his partners (if such exist) get what they are supposed to get in a counter objection. Obviously such a maximum exists, and $z_2 < x_2$, because $x_1 \in J$. Choose ε such that

$$-x_1 \leq \varepsilon < \min(\delta, y_1 - x_1, x_2 - z_2); \quad (7.7)$$

then $x_1 + \varepsilon$ will also belong to J . Indeed, (7.6) will be coalitionally rational; $(y; \mathcal{C})$ will remain an objection which is justified.

Thus, if $x_1 \in J$, then so are all the points on the interval $[0, x_1 + \varepsilon]$.

THEOREM 7.2 Let Γ be an n -person game, in which 12 is a permissible coalition and all the permissible coalitions are 1, 2 and 3-person coalitions; then, if (x, \mathcal{B}^0) is a c.r.p.c., there exists a c.r.p.c.

$$(\xi_1, \xi_2, x_3, x_4, \dots, x_n; 12, B_2, \dots, B_m) \quad (7.8)$$

such that neither player 1 nor player 2 has any justified objection. Here $\mathcal{B}^0 \equiv 12, B_2, \dots, B_m$.

Proof We proved in Theorem 7.1 that the numbers x_1 , for which player 1 has a justified objection, form an open set T_1 with respect to $[0, v(12)]$. Similarly, the numbers x_1 for which player 2 has a justified objection form an open set T_2 with respect to the same interval (x_3, \dots, x_n remain fixed). We shall show that T_1 and T_2 are disjoint, from which it will follow that there is a point ξ_1 in $[0, v(12)]$, which is neither in T_1 , nor in T_2 , and therefore (7.8) will satisfy the requirements. (None of the sets is the closed interval because $v(12) \notin T_1, 0 \notin T_2$.)

Indeed, suppose that

$$(\sigma_1, \sigma_2, x_3, \dots, x_n; 12, B_2, \dots, B_m) \quad (7.9)$$

is a c.r.p.c. in which both players have justified objections. Player 1, in his objection, must join a coalition C which contains more than one person and does not contain player 2. Similarly, player 2 must join, in his objection, a coalition D which consists of more than one person and does not contain player 1. If $C \cap D = \phi$, then player 2's objection can serve as a counter objection for player 1's objection, the latter being therefore not justified. If $C \cap D \equiv E = \phi$, then E contains one or two members. Without loss of generality, we can assume that the total amount that the players in E got in player 2's objection was not less than what they got from player 1's objection. If E contains one member, player 2's objection is a counter objection to player 1's objection. If E contains two members, this is not always true, because in order to counter object, player 2 has to modify, perhaps, his payments to the members of E . By doing so, a pay-

off configuration may result, which is not coalitionally rational; i.e., one player in E and player 2 can now make more by together forming a coalition. But if this is the case, then this coalition can serve in a counter objection; e.g., by player 2 taking σ_2 for himself and letting the other player get the rest.

Remark The theorem fails to hold if we remove the restriction on the the number of the players in the permissible coalitions. A counterexample is provided in Example 7.1.

Application Suppose that each one of two men got a license to build a gasoline station. Each one considers the possibility of taking at most two partners. They expect various profits from the corresponding possible coalitions. The other partners do not have licenses. Of course, the two men consider also their joint coalition. Under these assumptions, Theorem 7.2 says that the coalition of the two licensees is represented in the bargaining set.

8 The 4-Person Game in Which Only 1- and 2-Person Coalitions Are Permissible

The inequalities that determine under what condition is $(x_1, x_2, x_3, x_4; 12, 34)$ in \mathfrak{M} , for the game

$$\begin{cases} v(1) = v(2) = v(3) = v(4) = 0, & v(12) = a, & v(23) = b, & v(34) = c, \\ v(13) = d, & v(24) = e, & v(14) = f, & a, b, c, d, e, f \geq 0 \end{cases} \quad (8.1)$$

are given in Appendix 2.

Theorem 7.2 ensures that any partition which contains only one 2-person coalition is represented in \mathfrak{M} . We shall now study the case of partitions into two couples. Our object is to prove that some such partitions appear in \mathfrak{M} . It turns out that this can be proved even if we limit ourselves to *maximal* partitions, i.e., to those partitions in which the sum of the values of the coalitions is maximal. This restriction helps us by reducing the number of inequalities which need to be examined.

THEOREM 8.1 Let Γ be the game (8.1), where

$$a + c \geq d + e, \quad a + c \geq b + f; \quad (8.2)$$

then there always exists a p.c. $(x_1, x_2, x_3, x_4; 12, 34)$ in the bargaining set \mathfrak{M} .

Proof We omit the calculations, but state the various cases.

Case A. If

$$a \leq b + d, \quad b \leq a + d, \quad d \leq a + b, \quad 2c \geq b + d - a, \quad (8.3)$$

then

$$\left(\frac{a+d-b}{2}, \frac{a+b-d}{2}, \frac{b+d-a}{2}, \frac{2c+a-b-d}{2}; 12, 34 \right) \in \mathfrak{M}. \quad (8.4)$$

If

$$a \leq b + d, \quad b \leq a + d, \quad d \leq a + b, \quad 2c < b + d - a, \quad (8.5)$$

then

$$\left(\frac{a+d-b}{2}, \frac{a+b-d}{2}, c, 0; 12, 34 \right) \in \mathfrak{M}. \quad (8.6)$$

Case B. If

$$a > b + d, \quad c > d + f, \quad (8.7)$$

then

$$(d, a - d, 0, c; 12, 34) \in \mathfrak{M}. \quad (8.8)$$

Case C. If

$$a > b + d, \quad f > c + d, \quad b + c \geq e, \quad (8.9)$$

then

$$(f - c, a + c - f, 0, c; 12, 34) \in \mathfrak{M}. \quad (8.10)$$

(Indeed, (8.9) and (8.2) imply $c \geq e - b \geq e - (a + c - f)$, hence $a + 2c \geq e + f$. The rest follows directly.) If

$$a > b + d, \quad f > c + d, \quad e > b + c, \quad (8.11)$$

then

$$\left(\frac{a+f-e}{2}, \frac{a+e-f}{2}, 0, c; 12, 34 \right) \in \mathfrak{M}. \quad (8.12)$$

(Indeed, (8.2) and (8.11) imply $2d + e \leq d + a + c < a + f$. Also $2b + f \leq b + a + c < a + e$.)

Case D. If

$$a > b + d, \quad d > f + c, \quad b + c \geq e, \quad (8.13)$$

then

$$(a - b, b, 0, c; 12, 34) \in \mathfrak{M}. \quad (8.14)$$

If

$$a > b + d, \quad d > f + c, \quad e > b + c, \quad (8.15)$$

then

$$(a + c - e, e - c, 0, c; 12, 34) \in \mathfrak{M}. \quad (8.16)$$

Case E. If

$$d > a + b, \quad d > c + f, \quad (8.17)$$

then

$$(a, 0, d - a, c + a - d; 12, 34) \in \mathfrak{M}. \quad (8.18)$$

All other cases are either not maximal partitions, or they can be reduced to these cases by permuting the players:

$$1 \leftrightarrow 3, 2 \leftrightarrow 4. \quad (8.19)$$

9 The Restricted Bargaining Set

In a given game there are in general many stable p.c.'s. Though we do not possess a criterion for choosing between them, there are cases in which it is clear that some p.c.'s in \mathfrak{M} are "better" than others. We therefore suggest that the latter should be deleted from \mathfrak{M} , thus giving rise to the restricted bargaining set \mathfrak{M}^* .

A p.c. $(x; \mathcal{B})$ in \mathfrak{M} should be deleted if one of the following cases occurs:

(i) There exists in \mathfrak{M} a p.c. $(x^*; \mathcal{B}^*)$ with

$$x_i^* > x_i; \quad i = 1, 2, \dots, n. \quad (9.1)$$

(ii) There exists in \mathfrak{M} a p.c. $(x^{**}; \mathcal{B}^{**})$, where the coalitions in \mathcal{B}^{**} are unions of coalitions in \mathcal{B} , such that

$$x_i^{**} > x_i \quad (9.2)$$

for all the players i which belong to a union of more than one coalition of \mathcal{B} and

$$x_i^{**} \geq x_i \quad (9.3)$$

for all the other players.

One sees that in the examples given in the previous sections, only those coalitions which have relatively big values (if such exist), will appear in \mathfrak{M}^* .

10 Possible Modifications

Inasmuch as our theory tries to cope with “reality,” it is flexible enough to allow for some modifications.

For instance, if players are faced with the game treated in Example 7.2, they may claim that the demand for stability is too strong. They would rather relax this demand and still gain something from the game.

One can then offer them the following definition of a bargaining set \mathfrak{M}_1 :

DEFINITION 10.1 A c.r.p.c. $(x; \mathcal{B})$ belongs to the bargaining set \mathfrak{M}_1 , if for any objection K against L , there is somebody in L who can counterobject.

According to this definition, each player in a coalition B_j which contains K , who does not belong to the partners of K , is required to be able to counterobject; but several such players may perhaps be unable to protect their shares simultaneously. Clearly, the resulting bargaining set \mathfrak{M}_1 includes \mathfrak{M} , since the number of sets which is required to counterobject is reduced. In this case, e.g., the players of the game treated in Example 7.2 may agree to

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 123, 4, 5, 6\right), \quad (10.1)$$

which belongs to \mathfrak{M}_1 .

In some other real-life cases, one can estimate and tell in advance which coalitions may object and which coalitions may counterobject. This leads to various bargaining sets and brings us to the circle of ideas surrounding ψ -stability. (See Luce and Raiffa [3], pp. 163–168, 174–176, 220–236.)

One may limit K to be always one-person and L to be the remaining members of the coalition, except for K 's partners. This type of stability of one against the rest, which generates a bargaining set \mathfrak{M}_2 , is still different from the stability demanded in \mathfrak{M} , as the following example shows:

Example 10.1 Consider the game:

$$\begin{aligned} n = 5, \quad v(i) = 0, \quad v(123) = 30, \quad v(24) = v(35) = 50, \\ v(1245) = v(1345) = 60. \end{aligned} \quad (10.2)$$

Let

$$(x; \mathcal{B}) \equiv (10, 10, 10, 0, 0; 123, 4, 5). \quad (10.3)$$

If $K = 1, 2$, or 3 , then the remaining players which belong to the same

coalition and are not among his partners can always counterobject; but the objection for $K = 23$,

$$(0, 11, 11, 39, 39; 1, 24, 35), \quad (10.4)$$

has no counter objection because player 1 cannot keep his profits. Thus $(x; \mathcal{B}) \in \mathfrak{M}_2$ but $\notin \mathfrak{M}$.

It is easy to show that $\mathfrak{M} \subset \mathfrak{M}_2$.

Sometimes people would like to feel safe not only within their coalitions but also from “outside” threats. It may happen, e.g., that several players from *various coalitions* will threaten together other people from these coalitions. A reasonable way to cope with this strong demand for stability would be to allow K and L to belong to several coalitions, provided that K and L are required to intersect the same coalitions. This will bring us to a bargaining set \mathfrak{M}_0 which is *included* in \mathfrak{M} . Let us remark that $\mathfrak{M}_0 = \mathfrak{M}$ for the 2- and 3-person games, as well as for the 4-person game with only 1-, 3-, and 4-person coalitions permissible. If $n = 4$, where 1- and 2-person coalitions are permissible, one has to replace the inequalities of Appendix 2 by those listed in Appendix 3. Fortunately, these inequalities are satisfied in all the examples given in Section 8, and therefore Theorem 8.1 is valid if one replaces \mathfrak{M} by \mathfrak{M}_0 .

Finally, we would like to question the assumption of coalitional rationality. If we drop this condition, we may arrive at negative values in the bargaining set, but this does not have to bother us, since $(0, 0, \dots, 0; 1, 2, \dots, n)$ will certainly remain in the bargaining set, and therefore we can *demand* that the *restricted* bargaining set will contain only individually rational p.c.’s. However, we shall show in Example 10.2 that the resulting restricted bargaining set may still contain non-coalitionally rational p.c.’s.

Example 10.2 Let Γ be the game

$$v(i) = 0, \quad v(12) = v(45) = v(46) = v(56) = v(123) = 30, \quad v(34) = 10. \quad (10.5)$$

In this game, the non-coalitionally rational p.c.

$$(10, 10, 10, 0, 15, 15; 123, 4, 56) \quad (10.6)$$

is stable if one drops the condition of coalitional rationality. In fact, it then belongs to the restricted bargaining set, since otherwise there exists a p.c. $(x; \mathcal{B})$ in the bargaining set with

$$\sum_{i=1}^6 x_i > 60. \quad (10.7)$$

This can only occur if the coalition 34 is formed. Since, in addition, $x_3 \geq 10$, $x_4 \geq 0$, player 3 gets 10. This is impossible because in this case, player 4 has a justified objection, due to the fact that $x_1 + x_2 = 30$.

We have thus shown that the restricted bargaining set may contain a non-coalitionally rational p.c., if this condition is dropped from the definition of the bargaining set.

11 Concluding Remarks

Perhaps the nearest to our theory is W. Vickrey's concept of self-policing patterns [8]. His objections—called “heretical imputations”—are similar to ours; however, his counter objections—named “penalizing policing imputations”—are quite different.

Both the heretical and the penalizing policing imputations are in Vickrey's case *imputations*, whereas this is not the case in our theory. His penalizing policing imputation insists that at least one member of the “heretical coalition” is punished, whereas we only demand that the set L will be able to hold on to its property. However, the main difference lies, perhaps, in the fact that Vickrey is looking for a *set* of imputations—“self-policing patterns”—which are stable as a whole,⁸ while our bargaining set consists of payoff configurations, each one of which is stable in itself.

It has been pointed out in Section 10 that if a lack of communication is known to exist, one can incorporate ideas from ψ -stability theory into ours. Both theories stress the dependence of an outcome on the coalition structure which actually forms. However, ψ -stability theory requires the payoffs to be imputations,⁹ whereas we require that the outcome satisfies (2.4). (See, e.g., R. D. Luce and H. Raiffa [3], p. 222.) The coalitional rationality requirement (2.5) is a special case of a ψ -stability requirement, if one requires that all the subsets of the coalitions in the coalition structure τ are values of $\psi(\tau)$. A similar requirement appears also in Milnor's class L of reasonable outcomes ([3], pp. 240–242).

In many practical situations, the characteristic function is not the best way to describe a game. It would rather be better to apply the “Thrall characteristic function” (see R. M. Thrall [7]), which associates with each *coalition structure* a value for each coalition appearing in that structure. One can try to define the concepts of objections and counter objections, for such cases, and it is possible to do so in various ways.

8. In particular, he looks for “strong solutions.”

9. Or at least feasible individually rational n -tuples ([3], p. 226).

It is also possible to apply the notions described in this paper to the Aumann-Peleg characteristic function for cooperative games without side payments [2], essentially without change.

Finally, we should like to point out that our theory gives in many cases answers similar to those appearing in classical theories. Thus, e.g., the bargaining set in the discrete case of the 3-person non-zero-sum game¹⁰ consists essentially of the “central” three points of the non-discriminatory von Neumann–Morgenstern solution (see [9], pp. 550–554), but does not contain the additional “wiggles” that occur in their solution. The bargaining set for the non-discrete case is essentially the core. This suggests a pattern in which the bargaining set forms the “central” or “intuitive” part of a von Neumann–Morgenstern solution, whereas the “complications” disappear.

Appendix 1

Let Γ be a 4-person game, the coalitions of which, and their values, are given by (6.1). In order that the pair $(x_1, x_2, x_3, 0; 123, 4)$ belong to the bargaining set \mathfrak{M} , it is necessary and sufficient that

$$0 \leq x_1, \quad 0 \leq x_3, \quad x_1 + x_3 \leq a, \quad x_2 = a - x_1 - x_3, \quad (\text{A1.1})$$

and that at least one inequality (or equality) in each of the following rows should be satisfied.

$x_1 + x_3 \geq c$	$2x_1 + x_3 \geq a + c - d$	$x_1 + x_3 = a$
$x_3 \leq a - b$	$x_3 - x_1 \leq d - b$	$x_3 = 0$
$x_1 \leq a - d$	$2x_1 + x_3 \leq a + c - d$	$x_1 = 0$
$x_3 \leq a - b$	$x_1 + 2x_3 \leq a + c - b$	$x_3 = 0$
$x_1 + x_3 \geq c$	$x_1 + 2x_3 \geq a + c - b$	$x_1 + x_3 = a$
$x_1 \leq a - d$	$x_3 - x_1 \geq d - b$	$x_1 = 0$

Appendix 2

Let Γ be a 4-person game, the coalitions of which, and their values, are given by (8.1). In order that the pair $(x_1, x_2, x_3, x_4; 12, 34)$ belong to the

10. Clearly, one has to modify the characteristic function in the obvious way so as to get superadditivity.

bargaining set \mathfrak{M} , it is necessary and sufficient that

$$0 \leq x_1 \leq a, \quad 0 \leq x_3 \leq c, \quad x_1 + x_2 = a, \quad x_3 + x_4 = c, \quad (\text{A2.1})$$

and that at least one inequality (or equality) in each row be satisfied.

If the partition 12, 34 is maximal (in the sense of (8.2)), the last column can be omitted.

$x_1 = a$	$x_1 + x_3 \geq d$	$2x_1 \geq a + d - b$	$x_1 + x_3 \geq a + c - e$
$x_1 = a$	$x_1 - x_3 \geq f - c$	$2x_1 \geq a + f - e$	$x_1 - x_3 \geq a - b$
$x_1 = 0$	$x_1 - x_3 \leq a - b$	$2x_1 \leq a + d - b$	$x_1 - x_3 \leq f - c$
$x_1 = 0$	$x_1 + x_3 \leq a + c - e$	$2x_1 \leq a + f - e$	$x_1 + x_3 \leq d$
$x_3 = c$	$x_1 + x_3 \geq d$	$2x_3 \geq c + d - f$	$x_1 + x_3 \geq a + c - e$
$x_3 = c$	$x_1 - x_3 \leq a - b$	$2x_3 \geq c + b - e$	$x_1 - x_3 \leq f - c$
$x_3 = 0$	$x_1 - x_3 \geq f - c$	$2x_3 \leq c + d - f$	$x_1 - x_3 \geq a - b$
$x_3 = 0$	$x_1 + x_3 \leq a + c - e$	$2x_3 \leq b + c - e$	$x_1 + x_3 \leq d$

Appendix 3

The following inequalities replace those given in Appendix 2, if one desires that $(x_1, x_2, x_3, x_4; 12, 34)$ shall belong to \mathfrak{M}_0 . (See Section 10.) Again, at least one inequality in each row should be satisfied, as well as those given in (A2.1).

$x_1 + x_3 \geq d$	$x_1 = a$ $2x_3 \geq c + d - f$	$x_3 = c$ $2x_1 \geq a + d - b$	$x_1 + x_3 \geq a + c - e$
$x_1 - x_3 \geq f - c$	$x_1 = a$ $2x_3 \leq c + d - f$	$x_3 = 0$ $2x_1 \geq a + f - e$	$x_3 - x_1 \leq b - a$
$x_1 - x_3 \leq a - b$	$x_1 = 0$ $2x_3 \geq b + c - e$	$x_3 = c$ $2x_1 \leq a + d - b$	$x_1 - x_3 \leq f - c$
$x_1 + x_3 \leq a + c - e$	$x_1 = 0$ $2x_3 \leq b + c - e$	$x_3 = 0$ $2x_1 \leq a + f - e$	$x_1 + x_3 \leq d$

Bibliography

- [1] Aumann, R. J. and Maschler, M., "An equilibrium theory for n -person cooperative games," *Notices American Math. Soc.*, Vol. 8 (1961), p. 261.

- [2] Aumann, R. J. and Peleg, B., "von Neumann-Morgenstern solutions to cooperative games without side payments," *Bull. Amer. Math. Soc.* 66 (1960), pp. 173–179 [Chapter 38].
- [3] Luce, R. D. and Raiffa, H., *Games and Decisions*, John Wiley and Sons, New York (1957).
- [4] Maschler, M., "Playing an n -person game: an experiment," in *Contributions to Experimental Economics 8, Coalition Forming Behavior*, H. Sauermann (ed.), J. C. B. Mohr (Paul Siebeck), Tübingen, 1978, pp. 231–328.
- [5] Maschler, M., "Stable payoff configurations for quota games," *Advances in Game Theory*, Annals of Mathematics Studies No. 52, Princeton University Press, Princeton (1964), pp. 477–499.
- [6] Maschler, M., " n -Person games with only 1, $n - 1$, and n -person permissible coalitions," *J. Math. Analysis and Applications*, Vol. 6 (1963), pp. 230–256.
- [7] Thrall, R. M., "Generalized characteristic functions for n -person games," *Recent Advances in Game Theory, Proceedings of a Princeton University Conference, October 4–6, 1961*, ed. by M. Maschler. Philadelphia: Ivy Curtis Press. (1962), pp. 157–160.
- [8] Vickrey, W., "Self-policing properties of certain imputation sets," *Contributions to the Theory of Games*, Vol. IV, Annals of Mathematics Studies, No. 40, Princeton University Press, Princeton (1959), pp. 213–246.
- [9] Von Neumann, J. and Morgenstern, O., *Theory of Games and Economic Behavior*, 2nd ed., Princeton University Press, Princeton (1947).