Shapes and sizes of Laplace eigenfunctions

Steve Zelditch
Northwestern University
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Quantum ergodicity and nodal sets of eigenfunctions

This talk is concerned with the ‘semi-classical’ behavior as $\lambda \to \infty$ of the *nodal sets* = *zero sets*

$$\mathcal{N}_\lambda = \{x; \varphi_j(x) = 0\} \subset M$$

of the eigenfunctions of the Laplacian

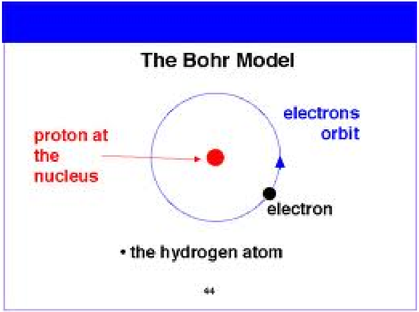
$$\Delta\varphi_j = \lambda_j^2 \varphi_j$$

on a Riemannian manifold (M, g) of dimension m.

We begin by recalling the motivation (in physics and mathematics) for studying eigenfunctions and their nodal sets.

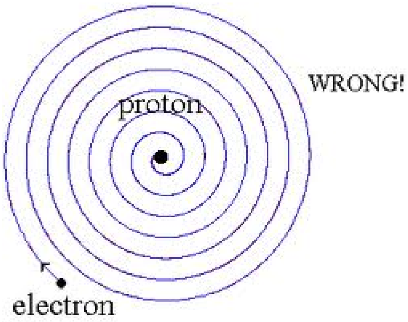
Visualizing an atom

Quantum mechanics resolves a puzzle about stability of atoms. Just before quantum mechanics, a hydrogen atom was roughly pictured as a 2-body planetary system, i.e. in terms of the classical Hamiltonian $H(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$ with $V = -\frac{1}{|x|}$.



Visualizing an atom

But that can't be right: the electron would radiate energy and spiral into the nucleus.



So Bohr proposed that the electron can only occupy special stable orbits.

Schrödinger equation

Schrödinger (Zurich, 1926) proposed the correct theory:

Quantisierung als Eigenwertproblem, Annalen der Physik (1926)

The energy states of the electron are modelled as eigenfunctions of the Schrödinger operator:

$$\hat{H}\varphi_j := (-\frac{\hbar^2}{2}\Delta + V)\varphi_j = E_j(\hbar)\varphi_j,$$

where $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$ is the Laplacian and V is the potential, a multiplication operator on L^2. Here $\hbar$ is Planck's constant. We let $\{\varphi_j\}$ denote an orthonormal basis (ONB) of eigenfunctions.

Stationary states

Quantum mechanics replaces classical mechanics with linear algebra (an eigenvalue problem). The time evolution of an energy state is given by

$$U_\hbar(t)\varphi_j = e^{-i\frac{t}{\hbar}(-\frac{\hbar^2}{2}\Delta + V)}\varphi_j = e^{-i\frac{tE_j(\hbar)}{\hbar}}\varphi_j.$$

The only *observable quantities* are the the modulus square $|\varphi_j(x)|^2 dx$ (the probability density of finding the particle at x) and matrix elements

$$\langle A\varphi_j, \varphi_j\rangle$$

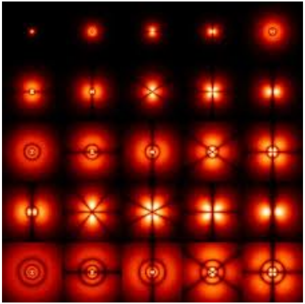
of observables (A is a self adjoint operator). The factors of $e^{-i\frac{tE_j(\hbar)}{\hbar}}$ cancel and so the particle evolves as if "stationary".

How to picture stationary states?

Quantum mechanics resolved the puzzle of how the electron can be moving and stationary at the same time. But it also replaced the geometric (classical mecahnical) Bohr model of classical orbits with eigenfunctions

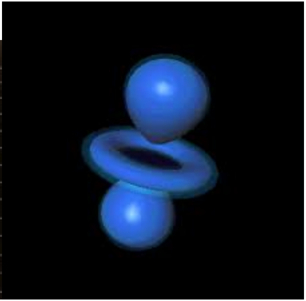
$$\hat{H}\varphi_j := (-\frac{\hbar^2}{2}\Delta + V)\varphi_j = E_j(\hbar)\varphi_j.$$

How can we picture eigenfunctions, i.e. stationary states of atoms?



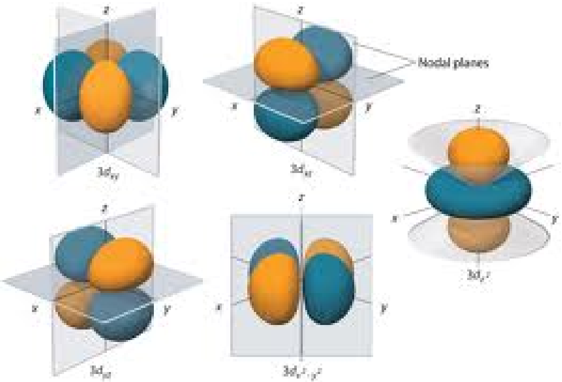
Intensity plots and excursion sets

One vivid kind of picture of (hydrogen) atom is an *intensity plot* which darkens in the regions where $|\varphi_j(x)|^2$ is large (most probable locations).

Nodal plots

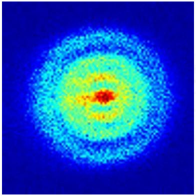
At the opposite are plots of the nodal hypersurfaces: the zero set $\mathcal{N}_j = \{x : \varphi_j(x, \hbar) = 0\}$.

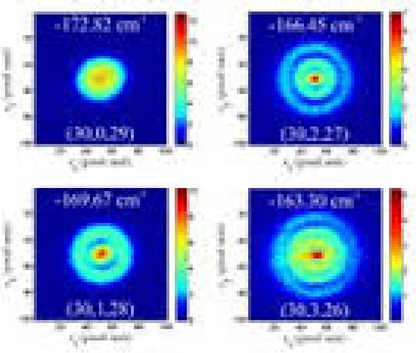


These are the points where the probability (density) of the particle's position vanishes.

Problems: how large is the nodal set in different regions? How is it

Experimental view of nodal sets of hydrogen: Stodoina





Vibrating string

Nodal sets are also important for vibrating strings, drums, membranes...the nodal set consists of the points where a vibrating membrane is stationary. In dimension 1 we are dealing with eigenfunctions $\varphi'' = -\lambda\varphi$ with $\varphi(0) = \varphi(L) = 0$, i.e. $\varphi(x) = \sin \frac{n\pi x}{L}$. The zeros are called nodes. Anti-nodes are the local maxima and minima.

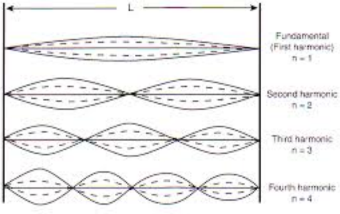
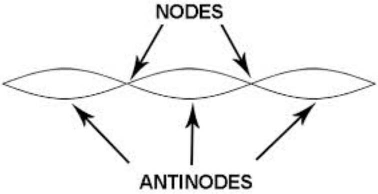


Figure 5

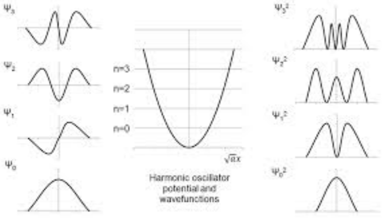


Sturm Liouville

More generally, one may study the real or complex zeros of one-dimensional Sturm-Liouville equations

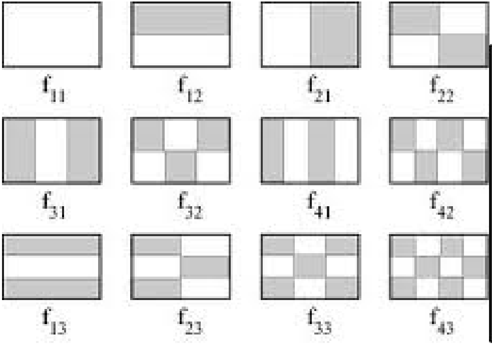
$$(-\hbar^2 \frac{d^2}{dx^2} + V(x))\psi(x) = E(\hbar)\psi(x), \quad x \in \mathbb{R},$$

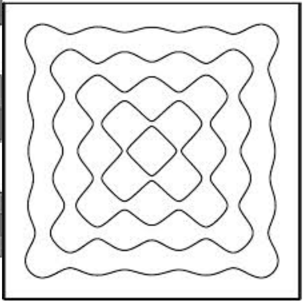
on all of $\mathbb{R}$ or on a finite interval. There are many classical results on the real zeros and some recent results on complex zeros (Hezari, Eremenko-Gabrielov-Shapiro). Below are graphics of Harmonic oscillatory eigenfunctions, $V = x^2$.



Higher dimensions; separation of variables

In higher dimensions, one would like to visualize modes by their nodes, but the nodal set is usually complicated. The only simple case is when one can separate variables and write eigenfunctions as products, $\psi(x, y) = f(x)g(y)$ of 1-dimensional functions. The system is then *completely integrable* , and the nodal sets form checkerboard patterns. If one take linear combinations, the checkerboard breaks up.





Goals

The goal is to describe asymptotics of nodal sets (and critical point sets – anti-nodes) on general Riemannian manifolds and to relate them to the dynamics of the geodesic flow. It is not obvious that any should exist! For simplicity we resrict to Laplacians (i.e. $V = 0$). We need to define the terms:

- Eigenfunctions of the Laplacian of a Riemannian manifold (M, g).
- Geodesic flow on the cotangent bundle T^*M of a Riemannian manifold (M, g).
- Nodal domains. Number of nodal domains.
- Distribution of nodal sets.

Nodal sets of eigenfunctions

Let (M, g) be a compact Riemannian manifold and let

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

be its Laplace operator.
Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

The NODAL SET of φ_j is its zero set:

$$\mathcal{N}_{\varphi_j} = \{x : \varphi_j(x) = 0\}.$$

Relation to classical mechanics

The Bohr model proposed a close relation between the quantum mechanics of a hydrogen atom and the classical mechanics of the corresponding classical Hamiltonian $H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x)$.

Can we truly use the classical mechanics to analyze shapes and sizes of quantum eigenstates, i.e. $||\varphi_j||_{L^p}$ or nodal sets $\mathcal{N}_j$? We would like to relate the quantum picture as $\hbar \to 0$ to the classical one.

In particular, what can we *prove* about nodal sets when the geodesic flow is ergodic ("chaotic").

Nodal domains

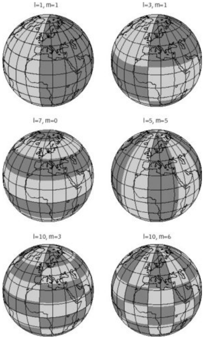
The nodal domains partition M into disjoint open sets:

$$M \backslash \mathcal{N}_{\varphi_\lambda} = \bigcup_{j=1}^{\mu(\varphi)} \Omega_j.$$

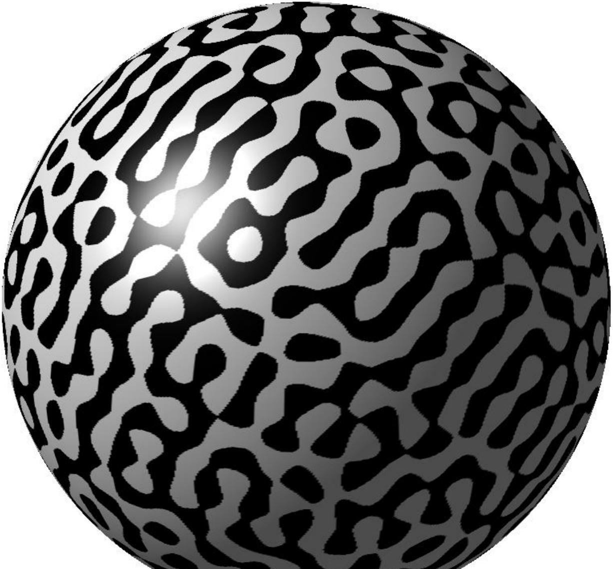
When 0 is a regular value of φ_λ the level sets are smooth curves.

When 0 is a singular value, the nodal set is a singular (self-intersecting) curve.

Nodal domains for $\Re Y_m^\ell$ spherical harmonics: geodesic flow integrable: Eigenfunctions coming from separation of variables



Degree 40 spherical harmonic



Some known results and conjectures

- There exist (M, g) and sequences $\varphi_{\lambda_{j_k}}, \lambda_{j_k} \to \infty$, with a uniformly bounded number of nodal domains: $N(\varphi_{\lambda_{j_k}}) \leq 3$ on the standard sphere (Hans Lewy), and ≤ 10 for some metrics on the 2-torus (Jakobson-Nadirashvili). Hence, $N(\varphi_{\lambda_{j_k}})$ does not have to grow to infinity.

- Conjecture: for any g there exists some sequence of eigenfunctions such that $N(\varphi_{\lambda_{j_k}}) \to \infty$.

Distribution of nodal hypersurfaces

How do nodal hypersurfaces wind around on M.?
We put the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$ to consider the nodal set as a *current of integration* Z_{φ_j}]: for $f \in C(M)$ we put

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}.$$

Problems:

- How does $\langle [Z_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.
- If $U \subset M$ is a nice open set, find the total hypersurface volume $\mathcal{H}^{n-1}(Z_{\varphi_j} \cap U)$ as $\lambda_j \to \infty$.
- How does it reflect dynamics of the geodesic flow?

Physics conjecture on real nodal hypersurface: ergodic case

Conjecture

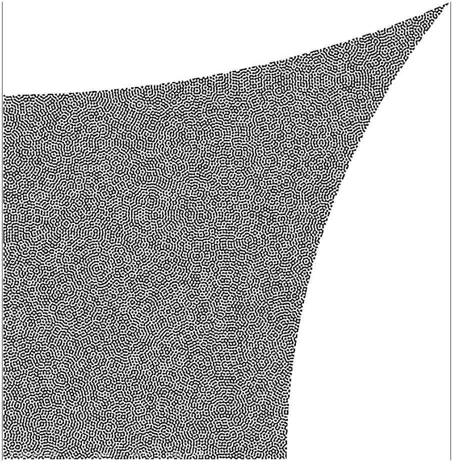
Let (M, g) be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,

$$\frac{1}{\lambda_j}\langle [Z_{\varphi_j}], f\rangle \sim \frac{1}{Vol(M,g)} \int_M f dVol_g.$$

Evidence: it follows from the "random wave model", i.e. the conjecture that eigenfunctions in the ergodic case resemble Gaussian random waves of fixed frequency.

Ergodic billiards

We expect the dynamics of the geodesic flow to have an important impact on the number of nodal domains. In the case of chaotic geodesic flow, we expect nodal domains to be random.



Two problems on nodal sets

We now present results on two problems:

1. How many nodal domains does an eigenfunction have? Let $N(\varphi_j)$ be the number of nodal domains of the jth eigenfunction (T. Hoffman-Ostenhof). Does $N(\varphi_j) \to \infty$ as $\lambda_j \to \infty$? I.e. do there always exist sequences of eigenfunctions φ_{j_k} so that $N(\varphi_{j_k}) \to \infty$

2. How are nodal sets distributed on M? Do they become uniformly distributed?

We give some results when the geodesic flow is ergodic.

Number of nodal domains

Suppose $\{\varphi_{\lambda_n}\}$ is an orthonormal basis of eigenfunctions with increasing eigenvalue: $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \uparrow \infty$. Let

$$N(\varphi_\lambda) = \#\text{Nodal domains}, \quad \nu(\varphi_\lambda) = \# \text{ components of } Z(\varphi_\lambda).$$

Courant nodal domain theorem: $N(\varphi_{\lambda_n}) \leq n$. In genus zero, the

number of components and the number of nodal domains are essentially the same if there are no singular points. In higher genus the relation is not so simple.

There are sharper upper bounds (Pleijel). But as examples of Lewy etc. show there are no universal lower bounds.

Of course, a lower bound on the number of nodal domains would give a lower bound on the number of critical points (for a Morse eigenfunction).

New result of Z with Junehyuk Jung

Let (M, J, σ) be a Riemann surface surface with an orienting-reversing involution σ and with $\mathrm{Fix}(\sigma)$ a separating set.

Let g be any negatively curved metric on M. We will show that for almost the entire sequence of even or odd eigenfunctions, the number of nodal domains tends to infinity.

The surfaces are special, but the argument works for any negatively curved metric. It only uses ergodicity of the geodesic flow.

In work in progress, we are proving the same result for bounded domains in $\mathbb{R}^2$ or the hyperbolic plane $\mathcal{H}^2$ with ergodic billiards.

Number of domains tends to infinity for almost all even/odd eigenfunctions

Theorem

Let (M, g) be a compact negatively curved C^∞ surface with an orientation-reversing isometric involution $\sigma : M \to M$ with $Fix(\sigma)$ separating. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of $L^2_{even}(Y)$, resp. $\{\psi_j\}$ of $L^2_{odd}(M)$, one can find a density 1 subset A of $\mathbb{N}$ such that

$$\lim_{\substack{j \to \infty \\ j \in A}} N(\varphi_j) = \infty,$$

resp.

$$\lim_{\substack{j \to \infty \\ j \in A}} N(\psi_j) = \infty,$$

For odd eigenfunctions, the conclusion holds as long as $Fix(\sigma) \neq \emptyset$.

Remarks

For a generic σ-invariant metric, the eigenvalues have multiplicity 1. Hence all eigenfunctions are either even or odd, and the parity restriction is not actually a restriction.

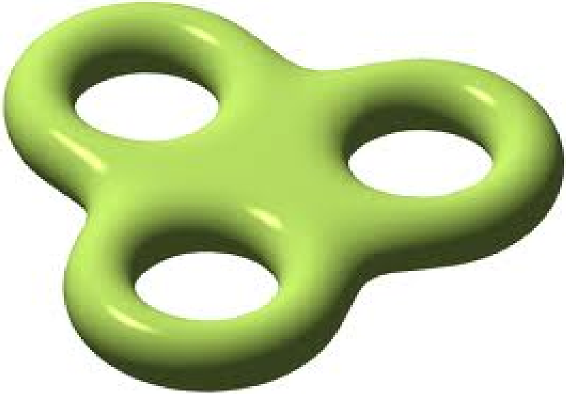
A density one subset $A \subset \mathbf{N}$ is one for which

$$\frac{1}{N}\#\{j \in A, j \leq N\} \to 1, \quad N \to \infty.$$

Hyperelliptic Riemann surface $g = 2$: Involution: top-bottom

As this picture indicates, the surfaces in question are complexifications of real algebraic curves. $Fix(\sigma)$ is the underlying real curve.

Hyperelliptic Riemann surface $g = 3$ top-bottom



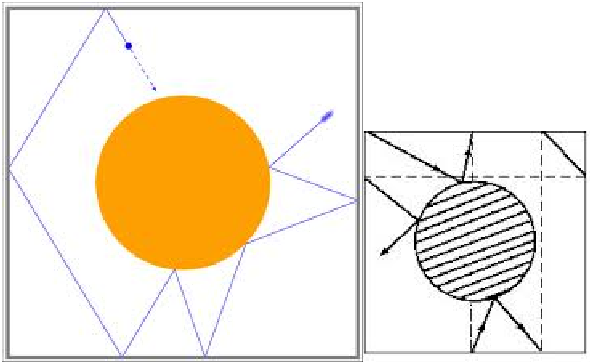
Same result Sinai billiards

Theorem

Let (X, g) be a surface with curvature $k \leq 0$ and let D be a small disc in X. Remove the disc to obtain a Sinai-Lorentz billiard $M = X \backslash D$. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of eigenfunctions, one can find a density 1 subset A of $\mathbb{N}$ such that

$$\lim_{\substack{j \to \infty \\ j \in A}} N(\varphi_j) = \infty,$$

Stadium: Ergodic billiards



Ghosh-Reznikov-Sarnak (2013)

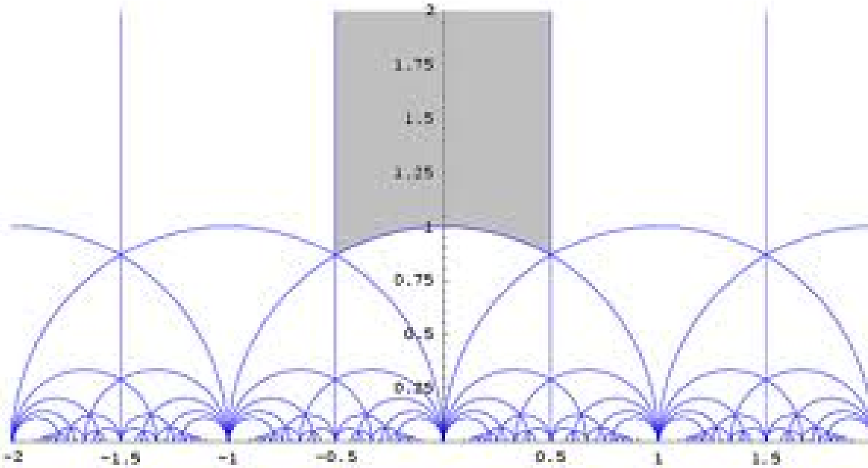
They give a power law lower bound for special eigenfunctions on a special (M, g) assuming the Lindelof hypothesis. The argument is the inspiration for our work:

Theorem

(GRS)
Let φ be an even Maass-Hecke L^2 eigenfunction on $\mathbb{X} = SL(2,\mathbb{Z})\backslash\mathbb{H}$. Denote the nodal domains which intersect a compact geodesic segment $\beta \subset \delta = \{iy \mid y > 0\}$ by $N^\beta(\varphi)$. Assume β is sufficiently long and assume the Lindelof Hypothesis for the Maass-Hecke L-functions. Then

$$N^\beta(\varphi) \gg_\epsilon \lambda_\varphi^{\frac{1}{24}-\epsilon}.$$

Modular surface and vertical geodesic



Equidistribution of nodal sets

The second result concerns the conjecture:

CONJECTURE

Let (M, g) be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,

$$\frac{1}{\lambda_j}\langle [Z_{\varphi_j}], f\rangle \sim \frac{1}{Vol(M,g)}\int_M f dVol_g.$$

We cannot prove or disprove it. But we can prove a positive result for

ANALYTIC CONTINUATIONS of EIGENFUNCTIONS $\varphi_j^{\mathbb{C}}$ to the complexification $M_{\mathbb{C}} \simeq T^*M$ when the geodesic flow is ergodic.

Equi-distribution of complex nodal sets in the ergodic case

Theorem

(Z, 2007) Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Then for all but a sparse subsequence of λ_j,

$$\frac{1}{\lambda_j} \int_{\mathcal{N}_{\varphi^{\mathbb{C}}_{\lambda_j}}} f \omega_g^{m-1} \to \frac{i}{\pi} \int_{M_\epsilon} f \bar{\partial}\partial\sqrt{\rho} \wedge \omega_g^{m-1}$$

Moreover (Z, 2013) Let γ be a geodesic satisfying a certain generic assymetry condition (postponed). Then for all but a sparse subsequence of λ_j, the intersection points $\zeta_k(\lambda_j) = t_k + i\tau_k$ of $\gamma_{\mathbb{C}} \cap \mathcal{N}_{\varphi^{\mathbb{C}}_{\lambda_j}}$ satisfy:

$$\frac{1}{\lambda_j} \sum_k f(\zeta_k(\lambda_j)) \to \int_{\mathbb{R}} f(t) dt.$$

Thus, the complex zeros condense on the real points of the geodesic and are uniformly distributed along it.